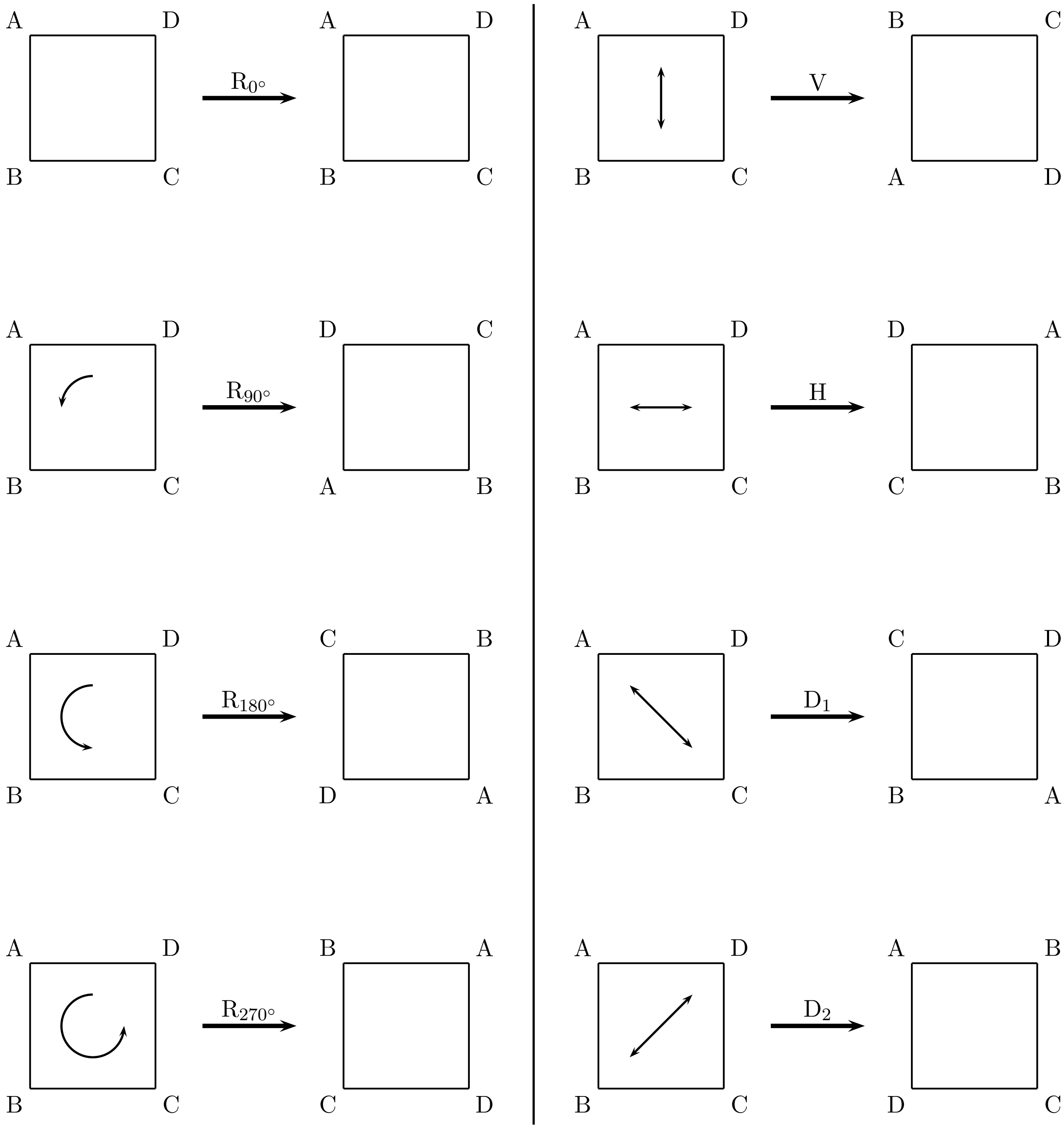
Symmetries of a Square



The Group of Symmetries of a Square

A *group* is a set of elements and an operation ($*$) that satisfy the following properties.

1. Closure. When two of the elements are combined using the operation, another element in the set is obtained.
2. Identity. There is a "do nothing" element in the set which, when combined with any other element using the operation, leaves that other element the same.
3. Inverses. Every element has an inverse element which combines with it using the operation to produce the Identity.
4. Associativity. For any three elements in the set, x, y, and z, $(x * y) * z = x * (y * z)$.

One simple example of a group is the set of real numbers with the operation of addition. The symmetries of a square form a group where the operation is *composition*, which in this case means applying one action followed by another.

Cayley Table for Composing Symmetries of a Square

		1st Operation							
		R_{0°	R_{90°	R_{180°	R_{270°	V	H	D_1	D_2
2nd Operation	R_{0°								
	R_{90°								
	R_{180°								
	R_{270°								
	V								
	H								
	D_1								
	D_2								

Group Actions

A *group action* consists of a group that acts on a set of objects via an operation $(*)$ that satisfies the following properties.

1. The identity element of the group takes every object to itself.
2. For all $g, h \in G$ and $x \in X$, $g * (h * x) = (gh) * x$.

Group Action Terminology

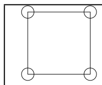
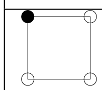
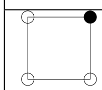
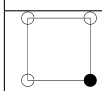
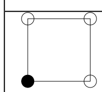
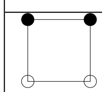
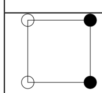
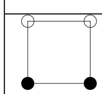
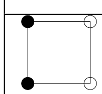
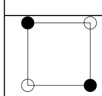
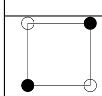
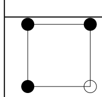
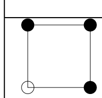
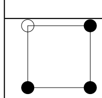
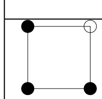
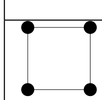
Fixing: We say that a group element *fixes* an object if it takes it to itself.

Stabilizer: The set of group elements that fix an object is called the *stabilizer* of the object.

Orbit: The set of objects that are reachable from a given object via group actions is called the *orbit* of the object.

Orbit-Stabilizer Theorem: Given a group action, the product of the size of the orbit of a given element x (written $G * x$) and the size of the stabilizer of x (written G_x) is equal to the size of the group (G). $|G * x| \cdot |G_x| = |G|$

Orbits and Stabilizers for the Group of Square Symmetries

	R_{0°	R_{90°	R_{180°	R_{270°	V	H	D_1	D_2
								
								
								
								
								
								
								
								
								
								
								
								
								
								
								
								

Counting Cube Colorings

Total Colorings in Chart (before applying symmetries):

Kind of Symmetry	Number of these symmetries	Number of colorings fixed	Total number of fixed colorings

Total Fixed Colorings:

Total Symmetry Group Elements:

Total Orbits:

Additional Challenge Questions

1. Suppose we want to construct cubes out of square faces that snap together and which come in four different colors. How many different cubes could be made?
2. Suppose we construct tetrahedra out of edge pieces that come in three different colors. How many different tetrahedra could be made?
3. A tic-tac-toe board is to be filled with X's and O's, with a total of five X's and four O's.
   (a) How many different ways are there to fill the board?
   (b) How many ways are there to fill the board if rotated versions of the same image do not count as different?
   (c) What if neither rotations nor reflections are counted as different?
4. A necklace has 4 black beads and 8 white beads. How many different necklaces are there matching this description?
5. Write your own question that could be solved using the techniques from this session!
6. Consider a rectangular grid with $m \times n$ squares. Each square is colored one of two colors. If you wish, you may choose a specific m and n to work with.
   (a) How many ways are there to do this up to 180° rotation?
   (b) How many ways are there to do this up to horizontal and vertical reflection symmetries?
7. Prove the Orbit-Stabilizer Theorem (either in general or for the specific context we explored here). Why is it that every orbit always has the same amount of X's?