Problems for San Jose Math Circle October 26, 2018

1 *Bug*. A bug is crawling on the coordinate plane from $(7,11)$ to $(-17,-3)$. The bug travels at constant speed one unit per second everywhere but quadrant II (negative x- and positive y-coordinates), where it travels at $\frac{1}{2}$ units per second. What path (with proof) should the bug take to complete her journey in minimal time?

2 *Blindfold*. There are 50 cards on a table top. You know that exactly 20 of them are face up, and the rest are face down. How can you separate the cards into two piles, each with the same number of face-up cards, *with your eyes closed*? No silly trickery used here; just simple mathematics.

3 *Annoying Brother*. There is an infinite supply of pool balls, which each have a positive integer written on it. For each integer label, there is an infinite supply of balls with that label.

You have a box which contains finitely many such balls. (For example, it may have six #3 balls and twelve #673 balls and a million #2 balls.) Your goal is to empty the box. You may remove any *single* ball you want at each turn. However, whenever you remove a ball, your little brother then adds more balls to your box with smaller labels. Your brother can put *any finite amount* of balls in, as long as they have a lower label number. For example, if you remove one #3 ball, your brother can replace it with 50 #2 balls and 2018 #1 balls.

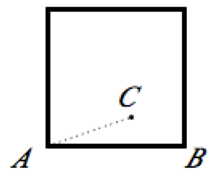
But If you remove a #1 ball, then your brother cannot mess things up, since there are no balls with lower numbers.

Is it possible to empty the box in a finite amount of time?

4 *Banquet*. People are seated around a circular table at a restaurant. The food is placed on a circular platform in the center of the table, and this circular platform can rotate (this is commonly found in Chinese restaurants that specialize in banquets). Each person ordered a different entrée, and it turns out that *no one* has the correct entrée in front of him. Is it always possible to rotate the platform so that at least *two* people will have the correct entrée?

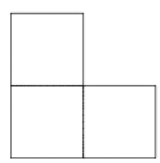
5 *Coloring*. Color the plane in 3 colors. Prove that no matter how the coloring is done, there are two points of the same color 1 unit apart.

6 * *Billiard ball*. A mathematical billiard ball (i.e., a point with zero radius) is shot from corner A of the square below as shown on a line with a slope of $17/90$. How many times will it bounce off a wall of the square before it returns to a corner, and which corner will it return to?



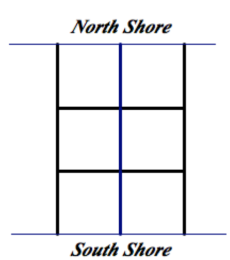
7 *A Divisive Game*. Consider the following two-person game between players A and B. The game starts with a positive integer S. A goes first, and the players alternate turns. On each turn, a legal move is to subtract any positive divisor of the current value as long as this divisor is strictly smaller than the current value. The winner is the last person to make a legal move (i.e., the person who leaves his or her opponent with the value of 1). For example, if $S = 12$, then A can take away 1, 2, 3, 4, or 6 on her first move. If she takes away 3, this leaves 9, and now B can take away 3 or 1, etc. Suppose that $S = 2018$. Find a winning strategy for one of the players, and prove that it works.

8 *Tiling with Ells*. An "ell" is an L-shaped tile made from three 1×1 squares, as seen below.



For what positive integers a, b is it possible to completely tile an $a \times b$ rectangle only using ells? For example, it is clear that you can tile a 2×3 rectangle with ells, but (draw a picture) you cannot tile a 3×3 with ells.

9 * *Bridges*. A system of 13 bridges, shown below, connects the north shore of a river to the south shore. For each bridge, there is a 50% probability that a protest march will block traffic across that bridge, and these probabilities are independent (imagine that each bridge flips a coin). What is the probability that it is possible to cross from one shore to the other?



10 *A Slightly Weird Function*. Let $f(n)$ be a function satisfying the following three conditions for all positive integers n:

(a) $f(n)$ is a positive integer,

(b) $f(n+1) > f(n)$,

(c) $f(f(n)) = 3n$.

Find $f(2018)$.

11 *Algebra?* How many distinct terms are there when

$$(1+x^7+x^{11})^{2018}$$

is multiplied out and simplified? You may use a computer to check your work, but your method must be simple and easy to do without a calculator.

12 *How Rich Can You Get?* Six boxes are arranged in a row, and initially, each contains one penny. The amount of pennies in the boxes changes according to the following two rules.

(a) **Rule A** says: if a box is non-empty, you can remove one penny from it, and then place two new pennies in the box immediately to the right. Notice that you cannot apply this rule to pennies in the rightmost box, since they have nowhere to go. For example, here are a few scenarios of the use of rule A:
[1|1|1|1|1|1] $\rightarrow$ [1|1|1|0|3|1] $\rightarrow$ [1|1|1|0|1|5].
For the last one, we actually used rule A twice.

(b) **Rule B** says: if a box contains at least one penny, and there are two boxes immediately to the right of it, you may remove a penny from this box, and then exchange the pennies in these two boxes. For example, if three boxes contain, in order from left to right, $6,0,3$, then after you apply rule B, the boxes now contain, in order from left to right: $5,3,0$. Note that this rule can only be applied to the first four boxes from the left, since you need two more boxes to the right of the box that you remove the penny from.

Here's an example showing the use of both rules:

[1|1|1|1|1|1] $\xrightarrow{A}$ [1|1|1|0|3|1] $\xrightarrow{B}$ [1|1|0|3|0|1].

Here's the question: start with one penny in each of the six boxes. You are allowed to apply rules A and B (one rule at a time, whichever rule that you wish, as long as you can apply the rule). How many pennies can you accumulate in the rightmost box?