TILING RECTANGLES

SIMON RUBINSTEIN-SALZEDO

1. A classic tiling problem

Question 1.1. *Suppose we tile a (large) rectangle with small rectangles, so that each small rectangle has at least one pair of sides with integer length. Does the large rectangle necessarily also have at least one pair of sides with integer length?*

By a tiling, we mean that the little rectangles completely cover the large one, and their interiors are pairwise disjoint, as in Figure 1.

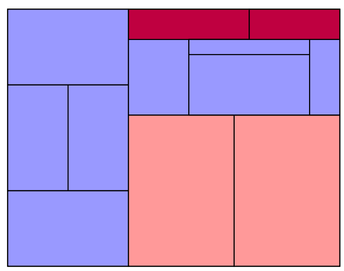

Figure 1. A tiling of a rectangle by smaller rectangles. Blue rectangles have horizontal edges of integer length; pink rectangles have vertical edges of integer length; maroon rectangles have all edges of integer length. Thus the horizontal edges of the large rectangle have integer length.

It is not obvious what the answer ought to be. But if you play around with it for a while, you will begin to suspect that the answer is "yes." And indeed, it is.

Theorem 1.2. *If we tile a large rectangle with small rectangles in such a way that each small rectangle has at least one pair of side with integer length, then the large rectangle also has at least one pair of sides with integer length.*

Proof 1. We color the plane in a checkerboard manner, as shown in Figure 2, so that each small square is $\frac{1}{2}$ by $\frac{1}{2}$. For any rectangle with sides parallel to the axes with lower left corner at $(0,0)$, if we superimpose it on the checkerboard coloring, then the black area is equal to the white area if and only if at least one pair of sides are integral. Otherwise, there is more black area. Furthermore, if we take any rectangle with sides parallel to the axes, even not based at $(0,0)$, then if it has at least one pair of integer sides, then it has the same amount of black and white area. So, does our large rectangle have the same amount of black and white area? Yes, because that is true of each small rectangle: the small rectangles all have at least one pair of integer sides and hence the same amount of black and white area. Thus the large rectangle has the same amount of black area and white area, and thus at least one pair of integer sides. ∎

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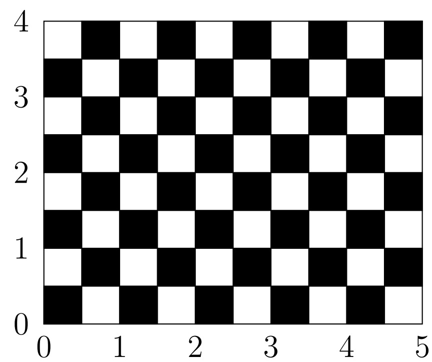

Figure 2. Checkerboard-coloring the plane

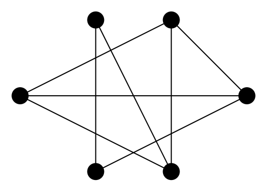

Figure 3. A graph with six vertices and eight edges

2. Making a graph

Let us now look at a completely different proof, based on constructing a useful graph. We will need the following terminology:

Definition 2.1. A graph is a pair $G = (V, E)$, where V is a set (finite, for us), and E is a collection of subsets of V of size 2. We call V the *vertices* and E the *edges*.

We usually draw a graph with a picture, where the vertices V are given by dots and the edges E are given by arcs between the vertices, as shown in Figure 3.

There is a lot that one can say about graphs, and we will do a fair amount of that over the next few weeks. But all we need at the moment is one definition and one lemma:

Definition 2.2. Let $G = (V, E)$ be a graph. For a vertex $v \in V$, the *degree* of v is the number of edges containing v. We write $\deg(v)$ for the degree of v.

Lemma 2.3. *If $G = (V, E)$ is any finite graph, then the sum of the degrees of the vertices is even.*

Proof. This is just about counting in two ways. We have

$$\sum_{v \in V} \deg(v) = \sum_{e \in E} 2 \equiv 0 \pmod 2,$$

as desired. ■

And now we're ready to give a new proof of the tiling theorem.

Proof 2. Let us place the rectangle so that the lower left corner is at $(0, 0)$. Let C denote the set of vertices of the small rectangles so that both the x and y coordinates are integers, and let T be the set of small rectangles. We construct a graph whose vertices are $C \cup T$ as

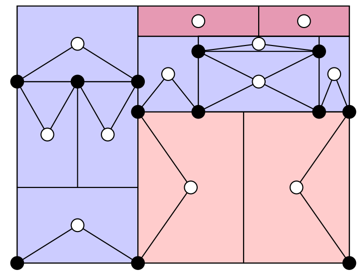

Figure 4. The black dots are C, and the white dots are T.

follows. The graph will be *bipartite*, so that all edges are between a vertex in C and a vertex in T. We put an edge between $c \in C$ and $t \in T$ if c is a vertex of t, as shown in Figure 4.

Now, observe that if $c \in C$ isn't a corner of the large rectangle, then it is a vertex of either 2 or 4 rectangles in T, so $\deg(c) \in \{2, 4\}$. However, if $c \in C$ is a corner of the large rectangle, then c is connected to a unique $t \in T$, because it's only touching a single rectangle, so $\deg(c) = 1$. In particular, $(0, 0)$ has a single neighbor. On the other hand, each $t \in T$ has degree 0, 2, or 4. Since the sum of the degrees is even, there must be an even number of vertices with odd degree, and the only vertices with odd degree are the corners of the large rectangle, *if* both of their coordinates are integers. Since $(0, 0)$ has odd degree, there must be another vertex with odd degree, i.e. another corner with integer coordinates. ■

3. Tiling squares with squares, and cubes with cubes

Is it possible to tile a square with squares? Yes, of course it is: if n is a positive integer, one can tile, for instance, an $n \times n$ square with n^2 1×1 squares. Or, even more stupidly, we can tile an $n \times n$ square with a single $n \times n$ square. But what if we require that there be at least two different squares in the dissection, and all the squares have to have *different* sidelengths? Such a dissection is known as a *perfect square dissection*. (Sometimes it is also known as *squaring the square*, in humorous analogue with the classical problem of squaring the circle.)

That is much trickier. One test case is the following famous identity:

$$\sum_{n=1}^{24} n^2 = 70^2,$$

which is one of the "coincidences" that makes the Leech lattice, a very dense and endlessly fascinating sphere packing in $\mathbb{R}^{24}$, possible. Is it possible to tile a 70×70 square with squares of sidelength $1, 2, 3, \ldots, 24$? Sadly, this turns out to be impossible, as can be shown with massive computation. However, in other cases, it can be done, as in Figure 5. The one in Figure 5 is the perfect square dissection with the smallest number of squares (other than 1).

However, one can find perfect square dissections of *rectangles* with far fewer squares. In fact, there is a perfect square dissection of a 33×32 rectangle with 9 squares; see problem 6. There is also an "almost-perfect" square dissection coming from Fibonacci numbers; see problem 7.

Now that we know that there are perfect square dissections of squares and rectangles, we can move up to higher dimensions and ask if there is a perfect cube dissection of a cube or a rectangular box.

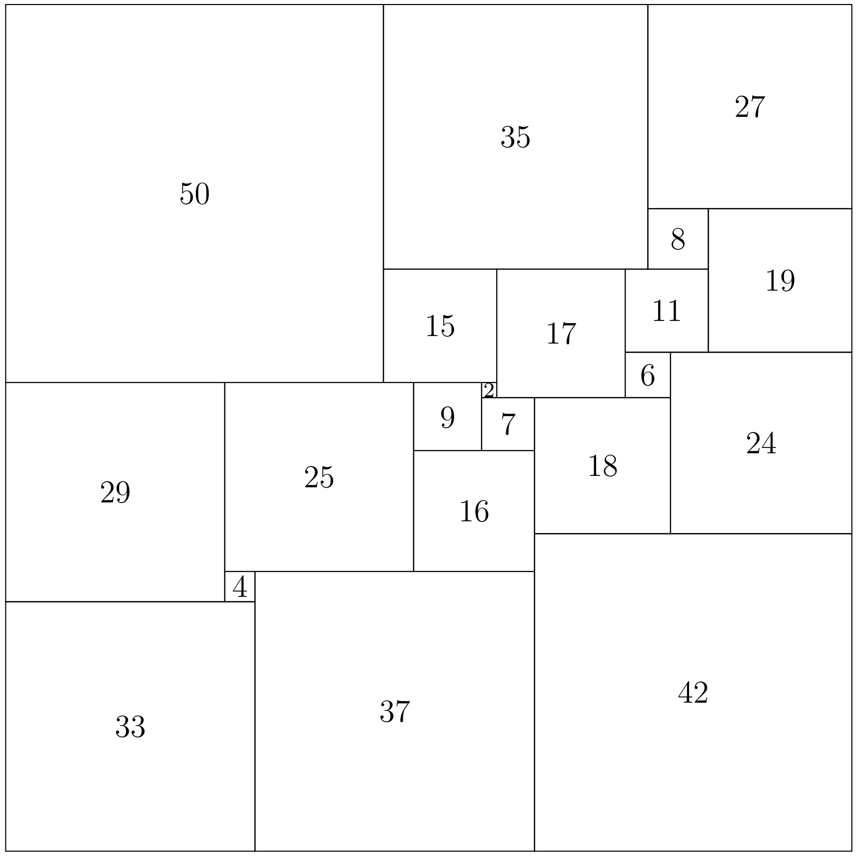

Figure 5. Tiling a 112×112 square with squares of sidelengths 2, 4, 6, 7, 8, 9, 11, 15, 16, 17, 18, 19, 24, 25, 27, 29, 33, 35, 37, 42, 50.

Theorem 3.1. *There is no perfect cube dissection of any rectangular box into a finite number of cubes.*

Proof. Suppose we have a perfect cube dissection of a rectangular box into a finite number of cubes. Let us look at the bottom face of the box. On this face, we have a perfect square dissection of a rectangle. Look at the smallest cube on this bottom face, and call it s_1. Since this cube is the smallest on the bottom face, it is surrounded by larger cubes, which are thus taller. Note that s_1 must be in the interior of the face, rather than on an edge or at a corner; see problem 11. Now, what happens at the top of s_1? Since all the cubes around it are larger, it must be tiled by a perfect square dissection of a square, all of which have smaller sidelength than s_1. Let s_2 be the cube directly above s_1 of minimal sidelength. Similarly, the top of s_2 be tiled by a perfect square dissection of a square, all of which have smaller sidelength than s_2, again because s_2 is in the interior of s_1, rather than at an edge or corner. Continue this way to get a sequence $s_1, s_2, s_3, \ldots$ of cubes. Thus the cube dissection consists of infinitely many cubes, contradicting our assumption about the finiteness of the dissection. ∎

4. Tiling a rectangle with congruent rectangles

Can we tile an $m \times n$ rectangle with $a \times b$ rectangles? We are allowed to use the $a \times b$ rectangles either as $a \times b$ rectangles or as $b \times a$ rectangles; that is, we are allowed to rotate them by $\frac{\pi}{2}$ if we choose. As usual, the answer is: sometimes. But which times? Let us see what can go wrong.

Example. Can we tile a 7×10 rectangle with 2×3 rectangles? Well, how many 2×3 rectangles would we need for such a tiling? We would need $\frac{7\times10}{2\times3} = \frac{35}{3}$ of them, and this number is not an integer. So, we cannot tile a 7×10 rectangle with 2×3 rectangles.

More generally, in order for a tiling to exist, it is necessary that mn be a multiple of ab. But this is not sufficient. (For instance, it is clear that we cannot tile a 1×12 rectangle with 2×3 rectangles, because we can't even fit a single 2×3 rectangle on the 1×12 board. This is an example of the next thing that can go wrong, but it is too simple for us to understand it at the correct level of generality.

Example. Can we tile a 17×28 rectangle with 4×7 rectangles? We don't run into the obstruction before, because 17×28 is a multiple of 4×7; all it tells us is that, if possible, it requires using 17 of the 4×7 rectangles. But that *might* be fine. So, let us imagine that we have a tiling of a 17×28 rectangle by 4×7 rectangles. Let us look at the top edge (one of the edges of length 17). The top edge must be broken down into a bunch of 4's and 7's. Let us say that the top edge contains s 4's and t 7's. Then we must have $4s + 7t = 17$. But it is easy to check that there are no nonnegative integers s and t so that $4s + 7t = 17$. Thus we cannot tile a 17×28 rectangle with 4×7 rectangles.

More generally, it is necessary that both the width m and height n can be covered with a's and b's, i.e. that both m and n can be written in the form $as + bt$, where s and t are nonnegative integers.

There is another obstruction, which is curiously related to the first problem we discussed this week.

Example. Can we tile a 10×15 rectangle with 1×6 rectangles? If so, we need $\frac{10\times15}{1\times6} = 30$ small rectangles. Furthermore, both 10 and 15 can be written as sums of 1's and 6's. However, we still cannot tile a 10×15 rectangle with 1×6 rectangles. In order to see this, let us suppose that we have a tiling, and then scale the entire picture by a factor of $\frac{1}{6}$, so that we would have a tiling of a $\frac{5}{3} \times \frac{5}{2}$ rectangle by $\frac{1}{6} \times 1$ rectangles. And now we can see why this cannot happen: because whenever we tile a rectangle with smaller rectangles so that all the smaller rectangles have at least one pair of integer sides, then the larger rectangle also has at least one pair of integer sides.

More generally, if we have a tiling of an $m \times n$ rectangle by $a \times b$ rectangles, then we can scale the picture by a factor of $\frac{1}{a}$ to obtain a tiling of an $\frac{m}{a} \times \frac{n}{a}$ rectangle by $1 \times \frac{b}{a}$ rectangles, so at least one of $\frac{m}{a}$ and $\frac{n}{a}$ must be an integer, i.e. at least one of m and n must be divisible by a. Similarly, if we scale by a factor of $\frac{1}{b}$, then we see that at least one of m and n must be divisible by b.

These are the only obstructions. Thus, we have the following theorem:

Theorem 4.1 (De Bruijn, Klarner)**.** *It is possible to tile an $m \times n$ rectangle with $a \times b$ rectangles if and only if the following conditions are satisfied:*

- *mn is a multiple of ab.*
- *Both m and n can be expressed in the form $as + bt$, where s and t are nonnegative integers.*

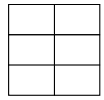

Figure 6. Tiling a square with rectangles similar to a 3×2 rectangle

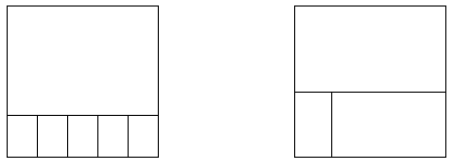

Figure 7. What does x have to be in each of these cases for all the tiles to be similar?

- *At least one of m and n is divisible by a, and at least one of m and n is divisible by b. (Note that this does* not *say that one of m and n is divisible by a, and the other one is divisible by b. That is* not *a necessary condition.)*

We leave the sufficiency of these conditions for problem 2; this amounts to showing that if the conditions hold, then there actually is a tiling. The way to do this is simply to explain how to construct the tiling.

5. Tiling a square with similar rectangles

A logical generalization of a tiling by congruent rectangles is a tiling by similar rectangles. So, when can we tile a rectangle by similar rectangles? Again, we allow ourselves to use the rectangles in either orientation. Now, if we can tile an $a \times b$ rectangle with rectangles similar to a $1 \times x$ rectangle, then we can tile an $a \times a$ square with rectangles similar to a $1 \times \frac{ax}{b}$ rectangle, simply by stretching or squishing the vertical direction. So, we may freely assume that we are trying to tile a square, say of sidelength 1, with rectangles similar to a $1 \times x$ rectangle.

First, if x is rational, then this is very easy. For example, we can tile a square with six rectangles similar to a 3×2 rectangle, as shown in Figure 6.

So, let us move on to the case where x is irrational. For example, we could consider the tiling in Figure 7 Left: what does the ratio of the sides have to be in order for this to be a tiling of the square?

Let's work it out! Let us say that the large rectangle at the top is $1 \times x$, and each of the small rectangles at the bottom are $\frac{1}{5} \times (1 - x)$. Then we must have

$$\frac{1}{x} = \frac{1-x}{1/5},$$

or $5x^2 - 5x + 1 = 0$. The solutions to this quadratic equation are $x = \frac{5 \pm \sqrt{5}}{10}$. The one we have drawn is the positive solution $\frac{5+\sqrt{5}}{10} \approx .7236$. The other solution, $\frac{5-\sqrt{5}}{10} \approx .2764$, also works, although the picture looks a bit different.

What about in Figure 7 Right? Let us say that the large rectangle is $1 \times x$. Then the medium rectangle is $\frac{1-x}{x} \times (1 - x)$, and the small rectangle is $x(1 - x) \times (1 - x)$, and also $(1 - \frac{1-x}{x}) \times (1 - x)$, or $\frac{2x-1}{x} \times (1 - x)$. Thus we must have $x(1 - x) = \frac{2x-1}{x}$, so $x^3 - x^2 + 2x - 1 = 0$.

This is an irreducible cubic equation, i.e. it does not factor over $\mathbb{Q}$. One root is $x \approx .5698$, which is the one we have drawn.

It is not so obvious, but for some values of x, even positive real numbers, we cannot tile a square with rectangles similar to a $1 \times x$ rectangle. By imagining mimicking what we did before, we may reason that x should be a root of a nonzero polynomial with integer coefficients; such an x is said to be an *algebraic* number. Not all numbers are algebraic; for instance, π and e are transcendental (not algebraic). But there are even positive algebraic numbers x such that the square cannot be tiled by rectangles similar to a $1 \times x$ rectangle.

Let us write $R(x)$ for a $1 \times x$ rectangle and T_u for the set of positive numbers x such that $R(x)$ can be tiled by rectangles similar to $R(u)$. We have the following:

- $u \in T_u$.
- $cu \in T_u$ for any positive rational number c.
- If $x \in T_u$, then $\frac{1}{x} \in T_u$.
- If $x, y \in T_u$, then $x + y \in T_u$.

Combining these, we find that, if $c_1, \dots, c_n$ are any positive rational numbers, then

$$c_1 u + \cfrac{1}{c_2 u + \cfrac{1}{\ddots + \cfrac{1}{c_n u}}} \in T_u.$$

We want to determine the values of u such that $1 \in T_u$.

We do not quite have the tools to prove it here (we need a bit of input from field theory), but here is the result:

Theorem 5.1 (Szekeres–Laczkovich)**.** *The following are equivalent:*

1. *The square can be tiled with similar copies of $R(u)$.*
2. *u is algebraic, and there is a nonzero polynomial $f(x)$ with integer coefficients such that $f(u) = 0$ and all roots of f have positive real part.*
3. *There are positive rational numbers $c_1, c_2, \dots, c_n$ such that*

$$c_1 u + \cfrac{1}{c_2 u + \cfrac{1}{\ddots + \cfrac{1}{c_n u}}} = 1.$$

As a consequence, we cannot tile the square with rectangles similar to $R(\sqrt[3]{2})$, since any polynomial with integer coefficients with $\sqrt[3]{2}$ as a root also has $\sqrt[3]{2} \times \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$ as a root, and this number has negative real part.

6. Tori

Rectangles are all very well, but they have an annoying flaw: their edges. The edges make rectangles not quite uniform. It is tempting to get rid of this anomaly by gluing together the opposite edges. If we glue together one pair of opposite edges, then we end up with a cylinder. If we then attempt to glue the other pair of opposite edges (which are now the top and bottom circles of the cylinder), then we have some difficulty in $\mathbb{R}^3$ if we make the rectangle out of paper, but if we're allowed to stretch it (for instance, if it's made out of

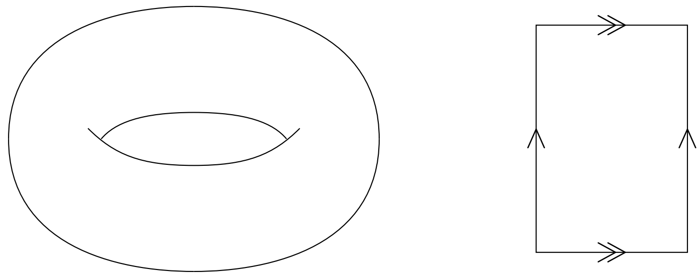

Figure 8. Two depictions of a torus

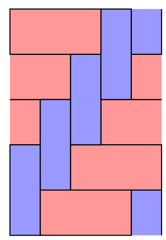

Figure 9. Tiling a $\frac{5}{3} \times \frac{5}{2}$ torus with $1 \times \frac{1}{2}$ and $\frac{1}{3} \times 1$ rectangles. The rectangles without boundaries continue on the other side of the rectangle, since these are glued together on the torus.

rubber), then we can freely glue them together, and we end up with a torus, as in Figure 8 Left.

But it's really easier to think of a torus as just being a rectangle with opposite edges glued together as shown in Figure 8 Right.

In order to tile a torus with rectangles, we use normal rectangles as before, except that they are allowed to move off one side and continue on along the opposite side.

We may wonder whether Theorem 1.2 is still true for a torus. But it is not, as we can see in Figure 9.

There is a version of Theorem 1.2 that holds for tori, but the statement and proof are a bit more complicated. We will not do the proof, but only state the theorem.

Theorem 6.1 (Robinson)**.** *Suppose an $a \times b$ torus is tiled with rectangles parallel to the sides of the torus. Suppose that each rectangle is designated as either an H-tile or a V-tile. Let G_H (resp. G_V) denote the set of sums and differences (with repetition) of the widths of the H-tiles (resp. heights of the V-tiles). Then at least one of the following is true:*

(1) $a \in G_H$,
(2) $b \in G_V$,
(3) *There exist relatively prime integers m, n so that $ma \in G_H$ and $nb \in G_V$.*

7. Problems

(1) Here are three classic problems that you should solve if you haven't seen them before:
  (a) Take an 8×8 square (e.g. a chessboard) and remove the top left and bottom right corner squares. Is it possible to tile the remaining region with dominoes, i.e. 1×2 and 2×1 rectangles?

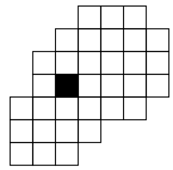

Figure 10. The black square is removed. Can the remaining region be tiled by dominoes?

   (b) An L-piece is a 2×2 square with one square removed. Show that if we take a $2^n \times 2^n$ board and remove any single 1×1 square from it, we can tile the remaining board with L-pieces.
   (c) Is it possible to tile the region in Figure 10 using dominoes? (There is a far-reaching generalization of this problem, known as *Hall's Marriage Theorem.*)

(2) Prove the sufficiency in Theorem 4.1 by explaining how to find a tiling of an $m \times n$ rectangle by $a \times b$ rectangles assuming the hypotheses are all satisfied.
(3) Find more proofs of Theorem 1.2.[1]
(4) Prove Theorem 1.2 with "integer" replaced with "rational." Which of the proofs work? Which ones don't? What about replacing "integer" with "algebraic"?
(5) Suppose that a box in $\mathbb{R}^n$ is tiled with small n-dimensional boxes so that each small n-dimensional box has at least k edges of integer length. Show that the large box also has at least k edges of integer length.
(6) Find a perfect square packing of a 33×32 rectangle using squares of sidelengths 1, 4, 7, 8, 9, 10, 14, 15, and 18.
(7) Find an "almost-perfect" square dissection of a rectangle coming from the Fibonacci numbers, using squares of sidelengths $1, 1, 2, 3, 5, 8, \dots, F_n$, where F_n is the n^{th} Fibonacci number. Generalize your construction to other Fibonacci-like sequences, i.e. sequences satisfying recurrence relations with nonnegative integer coefficients.
(8) Notice that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. Can you tile a 1×1 square using one rectangle of size $\frac{1}{n} \times \frac{1}{n+1}$ for each n?
(9) Is it possible to tile a square with pairwise noncongruent rectangles, all of the same area? What if we require the rectangles to have integer sidelength?
(10) Generalize Theorem 3.1 to higher dimensions, with proof. (Hint: This is easy once you know the result for cubes.)
(11) Prove that in any perfect square dissection of a rectangle R, the square of smallest sidelength does not lie on any edge of R.
(12) Given a perfect square dissection of a square, explain how to generate a perfect square dissection of the whole plane with squares of integer sidelength. How do the sidelengths grow in your construction? Can you find a perfect square dissection of the plane so that the sidelengths grow more slowly?
(13) Let n be a positive integer. Can you find a region made out of nonoverlapping squares of sidelength $1, 2, \dots, n$ (one of each) that tiles the plane?
(14) How many ways are there to tile a $2 \times n$ rectangle using 1×2 and 2×1 rectangles? What about a $3 \times n$ rectangle, assuming n is even?

[1]Stan Wagon has written a paper titled "Fourteen proofs of a result about tiling a rectangle," containing fourteen proofs of this theorem, as well as a discussion about how they generalize. See `http://www.latroika.com/mathoman/exos/wagon-integer-rectangle.pdf`.

Figure 11. An Aztec diamond of order 4 and a domino tiling

(15) An *Aztec diamond* of order n consists of all squares in a square lattice whose centers (x, y) satisfy $|x| + |y| \leq n$. (See Figure 11 for an Aztec diamond of order 4.) Find, with proof, the number of domino tilings of an Aztec diamond of order n, i.e. tilings by 1×2 and 2×1 rectangles.

Euler Circle, Palo Alto, CA 94306
E-mail address: simon@eulercircle.com