Monthly Contest 4
Due February 25, 2015

Instructions

This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problem's solution (not the total pages of all the solutions).

DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems.

You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may view any book or website (including forums) as long as you do not communicate the problem in any sort of manner (e.g. posting the problem statement or variations on a forum).

Problems

1. Find all ordered pairs of positive integers (x,y) such that $x^2 - y^2 < 20$

2. Prove that $x^2y + y^2z + z^2x \leq x^3 + y^3 + z^3$ for positive x, y, z.

3. Suppose we take all points (x,y) with integer x, y and assign them positive integral values with the following condition: the value of each point must be equal to the average of all the values of the points exactly one unit away from it. Prove that all points have the same value.

4. Take a regular pentagon with unit side length and draw all of its diagonals. Find the area of the newly-made smaller pentagon in the center.

5. Find all continuous real-valued functions f such that $f(x + 2y) = f(x) + 2f(y)$ for reals x, y.