Monthly Contest 4 Solutions

Here are the solutions for the fourth monthly contest of the 2013-2014 school year. The solutions listed below are not the only solutions. I encourage you to explore each problem and see if you can discover other approaches to the problems. Perhaps you may even find a solution more elegant than the ones listed!

Problem 1 Consider the equation $x^2 + kx + 2014 = 0$. Find the number of integers k for which the equation does not have a real solution.

Solution Recall that the quadratic formula states that the roots of this equation are $\frac{-k\pm\sqrt{k^2-4\times 2014}}{2}$. Note that real solutions require $k^2 - 4 \times 2014$ to be nonnegative. However, we want imaginary solutions so we need $k^2 - 4 \times 2014 < 0$ or $k^2 < 4 \times 2014$. One quick calculation finds that $|k| \le 89$ and there are 179 integers which satisfy this equation.

Problem 2 Prove that for any positive integer n, $1^4+2^4+3^4+\cdots+(n-1)^4+n^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.

Solution We will prove this by induction. For the base case, $n = 1$, we find that $1^4 = \frac{(1)(2)(3)(5)}{30}$ so we are finished. Now we perform the induction step. Let the formula be true for k. We will now prove that it is true for $k + 1$.

$$\begin{aligned}
1^4 + 2^4 + 3^4 + \cdots + (k-1)^4 + k^4 + (k+1)^4 &= (1^4 + 2^4 + 3^4 + \cdots + (k-1)^4 + k^4) + (k+1)^4 \\
&= \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4 \\
&= \frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30} \\
&= \frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)(3(k+1)^2+3(k+1)-1)}{30}
\end{aligned}$$

Thus, we have shown that the equation holds for $k + 1$ and we are done with the induction.

Problem 3 Let $\triangle ABC$ have inradius r and perimeter p. If R is the circumradius, show that $R \le \dfrac{p^2}{54r}$.

Solution Two identities of the triangle will help us solve this problem. These are that $S = \dfrac{rp}{2}$ and $4RS = abc$ where S represents the area and a, b, and c are the lengths of the triangle. We rearrange the second equation to get $R = \dfrac{abc}{4S}$. By using AM-GM, we find that $R \le \dfrac{(a+b+c)^3}{108S} = \dfrac{p^3}{54rp} = \dfrac{p^2}{54r}$.

Problem 4 Find all ordered pairs (m, n) such that $\binom{m}{n} = 2014$ where $\binom{a}{b}$ (pronounced "a choose b") represents the number of sets of b items one can pick out of a collection of a items.

Solution Two clear solutions are the ordered pairs (2014,1) and (2014,2013). We will now show there are no other solutions. The prime factorization of 2014 is $2 \times 19 \times 53$. We also know that $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ so a must be at least 53 so that the fraction has 53 as a prime factor. Note that if b is between 3 and $a-3$, then the fraction much too large with $a \geq 53$. Thus, the only other solutions must have $b = 2$ or $b = a - 2$. We only need to consider the case $b = 2$ since the other case is symmetric. $\binom{a}{2} = \frac{a(a-1)}{2}$. For this to equal 2014, we need $a(a-1) = 4028$, but this has no integer solutions. Thus, the only solutions to this equation are (2014,1) and (2014,2013).

Problem 5 The Cauchy-Schwartz Inequality states that

$$(a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2)(b_1^2 + b_2^2 + b_3^2 + \cdots + b_{n-1}^2 + b_n^2) \geq (a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_{n-1}b_{n-1} + a_nb_n)^2$$

for all real a_i and b_i. Prove that

$$\frac{a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2}{n}}$$

for all real a_i. (You are not required to use the Cauchy-Schwartz Inequality)

Solution We can substitute $b_i = 1$ for all i into the Cauchy-Schwartz Inequality. This gives us

$$(a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2)(1^2 + 1^2 + 1^2 + \cdots + 1^2 + 1^2) \geq (a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + \cdots + a_{n-1} \cdot 1 + a_n \cdot 1)^2$$

$$n(a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2) \geq (a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n)^2$$

$$\sqrt{n(a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2)} \geq a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

$$\sqrt{\frac{a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2}{n}} \geq \frac{a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n}{n}$$

And this is what we wished to prove.