Monthly Contest 4
Due March 12, 2014

Instructions

This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of problem's solution (not the total pages of all the solutions). DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems.

You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may consult any book that you wish.

Problems

1. Consider the equation $x^2 + kx + 2014 = 0$. Find the number of integers k for which the equation does not have a real solution.

2. Prove that for any positive integer n, $1^4 + 2^4 + 3^4 + \cdots + (n-1)^4 + n^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.

Hint: Induction works great for this problem.

3. Let $\triangle ABC$ have inradius r and perimeter p. If R is the circumradius, show that $R \leq \dfrac{p^2}{54r}$.

Hint: Look at 2013 - 2014 Monthly Contest 3 #3. Also, you may need to look up certain relations between area, inradius, circumradius, and the sides of the triangle.

4. Find all ordered pairs (m, n) such that $\dbinom{m}{n} = 2014$ where $\dbinom{a}{b}$ (pronounced "a choose b") represents the number of sets of b items one can pick out of a collection of a items.

5. The Cauchy-Schwartz Inequality states that

$$(a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2)(b_1^2 + b_2^2 + b_3^2 + \cdots + b_{n-1}^2 + b_n^2) \geq (a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_{n-1}b_{n-1} + a_nb_n)^2$$

for all real a_i and b_i. Prove that

$$\frac{a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2}{n}}$$

for all real a_i. (You are not required to use the Cauchy-Schwartz Inequality)