Monthly Contest 3 Solutions

Here are the solutions for the third monthly contest of the 2013-2014 school year. The solutions listed below are not the only solutions. I encourage you to explore each problem and see if you can discover other approaches to the problems. Perhaps you may even find a solution more elegant than the ones listed!

Problem 1 Find the sum of all possible units digits for $n!$ where n is a nonnegative integer.

Solution We know that $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, and $4! = 24$. Once $n \geq 5$, we have that the units digit is 0 since all the factorials after that will have a factor of 10 in them. Thus, the only possible units digits are 0, 1, 2, 4, and 6 so our answer is $0 + 1 + 2 + 4 + 6 = 13$.

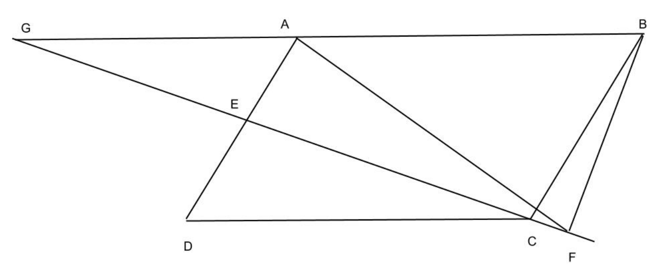
Problem 2 Find the remainder when $2^{2014} + 2014^2$ is divided by 13.

Solution 2014 has a remainder of 12 when divided by 13 so 2014^2 has a remainder of 1. Also, by Fermat's last theorem, we have that $2^{12} \equiv 1 \pmod{13}$. Since 2014 has a remainder of 10 when divided by 12, we have that $2^{2014} \equiv 2^{10} \equiv 10 \pmod{13}$. This shows us that $2^{2014} + 2014^2$ have a remainder of $1 + 10 = 11$ when divided by 13.

Problem 3 If a, b, and c are nonnegative numbers, show that $(a + b + c)^3 \geq 27abc$.

Solution From AM-GM, we have that $(a + b + c)^3 \geq (3\sqrt[3]{abc})^3 = 27abc$.

Problem 4 On parallelogram ABCD, point E lies on the midpoint of $\overline{AD}$. Point F is drawn on $\overline{CE}$ such that $\overline{BF}$ is perpendicular to $\overline{CE}$. Show that $\triangle ABF$ is isosceles.



Solution We extend $\overline{CE}$ through point E until it hits $\overleftrightarrow{AB}$. Call this intersection G. Since $\overline{AE} = \frac{1}{2}\overline{AD} = \frac{1}{2}\overline{CB}$ and $\overline{AD} \parallel \overline{BC}$, we have that $\overline{GA} = \overline{AB}$. We also have that $\triangle BFG$ is a right triangle, so $\overline{BG}$ is the diameter of its circumcircle. Thus, A is the center of this circle, so $\overline{AF}$ and $\overline{AB}$ are radii and must have equal length. Therefore, $\triangle ABF$ is isosceles.

Problem 5 One interesting property with Fibonacci numbers is that they follow Binet's formula. That is

$$F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

where F_n is the n^{th} Fibonacci number where $F_0 = 0$ and $F_1 = 1$. If $\lfloor x \rfloor$ is the greatest integer less than or equal to x, then show that

$$F_n = \left\lfloor \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n + 1) \right\rfloor$$

for $n \geq 0$.

Solution We use the fact that $a = \lfloor b \rfloor$ for integer a if and only if $0 \leq b - a < 1$. Thus, we want to show that

$$0 \leq (\frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n + 1)) - (\frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)) < 1$$

Cancelling out common terms, we get simplify the expression to

$$0 \leq \frac{1}{\sqrt{5}}(1 - (-(\frac{1-\sqrt{5}}{2})^n)) = \frac{1}{\sqrt{5}}(1 + (\frac{1-\sqrt{5}}{2})^n) < 1$$

Note that $|\frac{1-\sqrt{5}}{2}| < 1$ so $|(\frac{1-\sqrt{5}}{2})^n| \leq 1$ for all $n \geq 0$. From this, we deduce that $0 \leq 1 + (\frac{1-\sqrt{5}}{2})^n \leq 2$ and that $0 \leq \frac{1}{\sqrt{5}}(1 + (\frac{1-\sqrt{5}}{2})^n) < 1$. We have now proved that

$$F_n = \left\lfloor \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n + 1) \right\rfloor$$