Monthly Contest 3
Due February 5, 2014

Instructions

This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of problem's solution (not the total pages of all the solutions). DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems.

You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may consult any book that you wish.

Problems

1. Find the sum of all possible units digits for $n!$ where n is a nonnegative integer.

2. Find the remainder when $2^{2014} + 2014^2$ is divided by 13.

3. If a, b, and c are nonnegative numbers, show that $(a+b+c)^3 \geq 27abc$.

Hint: Arithmetic Mean-Geometric Mean Inequality

4. On parallelogram ABCD, point E lies on the midpoint of $\overline{AD}$. Point F is drawn on $\overline{CE}$ such that $\overline{BF}$ is perpendicular to $\overline{CE}$. Show that $\triangle ABF$ is isosceles.

Note: This is the corrected version of problem 4 from Monthly Contest 2.

5. One interesting property with Fibonacci numbers is that they follow Binet's formula. That is

$$F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

where F_n is the n^{th} Fibonacci number where $F_0 = 0$ and $F_1 = 1$. If $\lfloor x \rfloor$ is the greatest integer less than or equal to x, then show that

$$F_n = \left\lfloor \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n + 1) \right\rfloor$$

for $n \geq 0$.