Russian Math Circle Problems

Fall 2010

Instructions: Work as many problems as you can. Even if you can't solve a problem, try to learn as much as you can about it. Please write a complete solution to each problem you solve, as if you were entering it into a math contest and had no ability to explain it to the grader. This will help you make sure that you've thought of all the possibilities.

1. In the expression below, insert parentheses and/or signs $+$, $-$, $\times$, $/$ in order to get a true equation:
$$1/2 \quad 1/6 \quad 1/6030 = 2010$$

2. Solve the cryptogram: $SEND + MORE = MONEY$. Each letter represents a digit.

3. Find every integer which is equal twice the sum of its digits.

4. Paint every point of the plane using exactly three colors so that every line contains points of exactly two colors. How many ways can you think of to do this?

5. The numbers from 1 to 64, inclusive, are placed in an 8×8 grid, one number per square. If the sum of each row is the same, we call the arrangement BALANCED. If the sum of the rows are all different, we call the arrangement SILLY. Prove that the number of balanced arrangements is less than or equal to the number of silly arrangements.

6. In trapezoid $ABCD$ with BC parallel to AD, let $BC = 1000$ and $AD = 2008$. Let the angle A be $37°$, angle D be $53°$, and M and N be the midpoints of BC and AD, respectively. Find the length MN. (AIME, 2008, #5).

7. Find $ax^5 + by^5$ if the real numbers a, b, x and y satisfy:
$$\begin{aligned} ax + by &= 3 \\ ax^2 + by^2 &= 7 \\ ax^3 + by^3 &= 16 \\ ax^4 + by^4 &= 42 \end{aligned}$$
(AIME, 1990, #15)

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1. In the expression below, insert parentheses and/or signs $+$, $-$, $\times$, $/$ in order to get a true equation:
$$1/2 \quad 1/6 \quad 1/6030 = 2010$$
Solution:
$$(1/2 - 1/6)/(1/6030) = 2010$$

2. Solve the cryptogram: $SEND + MORE = MONEY$. Each letter represents a digit.

   Solution: $9567 + 1085 = 10642$.

3. Find every integer which is equal twice the sum of its digits.

   Solution: Let $(a_n a_{n-1} \ldots a_1)_{10}$ denote the decimal representtion of a number A so that
$$A = (a_n a_{n-1} \ldots a_1)_{10} = 10^{n-1} a_n + 10^{n-2} a_{n-1} + \cdots + 10a_2 + a_1.$$
We want to find all solutions such that
$$2(a_n + a_{n-1} + \cdots + a_1) = 10^{n-1} a_n + 10^{n-2} a_{n-1} + \cdots + 10a_2 + a_1,$$
and hence
$$a_1 = (10^{n-1} - 2)a_n + (10^{n-2} - 2)a_{n-1} + \cdots + (10^1 - 2)a_2,$$
if $n \geq 2$.

   If A is an n-digit number, either $n = 1$ and $a_1 = 0$ or $a_n \geq 1$. If $n \geq 3$ then $a_1 \geq (10^{n-1} - 2)a_n \geq 98$, which is a contradiction, so $n = 1$ or $n = 2$. If $n = 1$ then $A = a_1 = 2a_1$, so $a_1 = 0$. If $n = 2$ then $a_1 = 8a_2$ and the only possible solution is $A = 18$. Thus 0 and 18 are the only solutions.

4. Paint every point of the plane using exactly three colors so that every line contains points of exactly two colors. How many ways can you think of to do this?

   Solution: Here is one answer: On a white plane draw two blue intersecting lines and paint their point of intersection red.

   In fact, if we paint one point of the plane red, and consider all of the lines passing through it, as long as each is a constant white or blue color except for the red point and as long as at least two of the lines are white and at least two of the lines are blue, we have a solution.

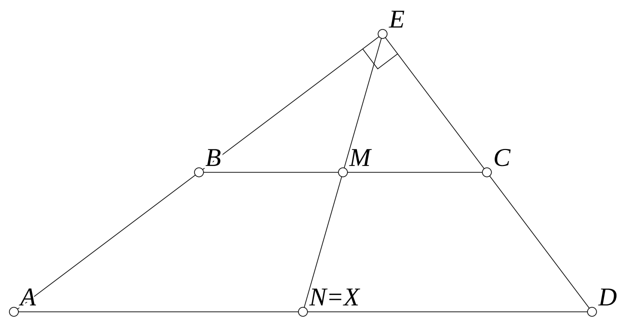
5. The numbers from 1 to 64, inclusive, are placed in an 8×8 grid, one number per square. If the sum of each row is the same, we call the arrangement BALANCED. If the sum of the rows are all different, we call the arrangement

SILLY. Prove that the number of balanced arrangements is less than or equal to the number of silly arrangements.

Solution: Suppose you have a balanced arrangement. Consider the elements of an arbitrary column, say the rightmost column. They are all different. Index them by row; call them $a_1, a_2, \ldots, a_8$. Now permute them by reversing their relative magnitudes. In other words, exchange the largest a_k with the smallest one; the second largest with the second smallest, etc. This will ensure that all rows now have different sums (in particular, the row that had the largest element in its rightmost column now has the least sum, while the row that had the smallest rightmost element now has the largest sum, etc.)

6. In trapezoid $ABCD$ with BC parallel to AD, let $BC = 1000$ and $AD = 2008$. Let the angle A be $37°$, angle D be $53°$, and M and N be the midpoints of BC and AD, respectively. Find the length MN. (AIME, 2008, #5).

   Solution: See the figure below:

   Extend AB and DC to their point of intersection E. Since $37+53 = 90$, triangle AED is a right triangle. Suppose the line EM intersects AD at a point X. Since triangles BEM and AEX are similar, $EM/BM = EN/AX$. Likewise, since triangles CEM and DEX are similar, $EM/MC = EN/DX$. But $BM = MC$, and hence $EN/AX = EN/DX$, so $AX = DX$, and therefore $X = N$.

   BC is a diameter of the circle through B, E, and C, so M is the center of this circle and thus $EM = BM = MC = 500$. Similarly, N is the center of the circle through A, E, and D, and hence $EN = AN = ND = 1004$. Thus $MN = EN - EM = 1004 - 500 = 504$.

7. Find $ax^5 + by^5$ if the real numbers a, b, x and y satisfy:

$$\begin{aligned} ax + by &= 3 \\ ax^2 + by^2 &= 7 \\ ax^3 + by^3 &= 16 \\ ax^4 + by^4 &= 42 \end{aligned}$$

   (AIME, 1990, #15)

Solution: Notice that:

$$\begin{aligned}(ax^2+by^2)(x+y) &= (ax^3+by^3)+(ax+by)xy\\(ax^3+by^3)(x+y) &= (ax^4+by^4)+(ax^2+by^2)xy\\(ax^4+by^4)(x+y) &= (ax^5+by^5)+(ax^4+by^4)xy\end{aligned}$$

Therefore:

$$\begin{aligned}7(x+y) &= 16+3xy\\16(x+y) &= 42+7xy\\42(x+y) &= (ax^5+by^5)+16xy\end{aligned}$$

Solving the first two equations for $(x+y)$ and xy we obtain $(x+y)=-14$ and $xy=-38$. Now the last equation becomes:

$$42(-14)=(ax^5+by^5)+16(-38).$$

Thus $ax^5+by^5=16(38)-42(13)=20$.