# Russian Math Circle Problems

November 10, 2010

Instructions: Work as many problems as you can. Even if you can't solve a problem, try to learn as much as you can about it. Please write a complete solution to each problem you solve, as if you were entering it into a math contest and had no ability to explain it to the grader. This will help you make sure that you've thought of all the possibilities.

1. Let p and q be prime numbers, and let n be a whole number that satisfy:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{pq} = \frac{1}{n}.$$

   Find all possible sets of values p, q and n.

2. Find all possible sets of colored beads with the property that you can make exactly two different necklaces using all the beads in your set.

3. Prove that the sum of the measures of the angles $A + B + C + D + E + F + G$ in the figure below is $180°$.
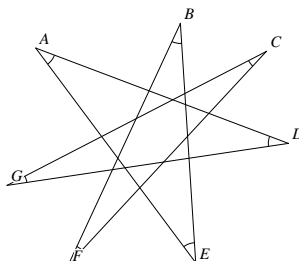

4. Let:
$$A = \{0, 1, 2, 3, \ldots, 3^k - 1\}.$$

   Show that you can choose 2^k of them such that none of the numbers you choose is equal to the arithmetic mean of two of the other numbers you chose.

5. A triangle is made of pennies with n pennies on a side and so that the tip is pointing up. What is the minimum number of pennies that must be moved to turn the triangle upside down? The figure below shows how this might be accomplished with two moves in the case $n = 3$:
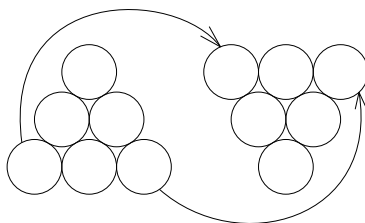
6. A circular track is 100 kilometers long. Ten identical cars are arranged on it equally-spaced, so that each is 10 kilometers from the nearest neighbor cars. Each car can drive 10 kilometers on one liter of gasoline, and initially the total amount of gasoline in all the tanks is 10 liters. Show that it is possible to choose one of the cars, drive it to the next car, put all the remaining gas into one of them, then continue driving to the next and the next, moving all the gas into one of the cars each time until you have made one complete loop and wind up where you started.

What if the cars are *not* equally-spaced at the beginning, but are arbitrarily-arranged, but still with 10 total liters of gas among them?