Russian Math Circle Problems

November 10, 2010

Instructions: Work as many problems as you can. Even if you can't solve a problem, try to learn as much as you can about it. Please write a complete solution to each problem you solve, as if you were entering it into a math contest and had no ability to explain it to the grader. This will help you make sure that you've thought of all the possibilities.

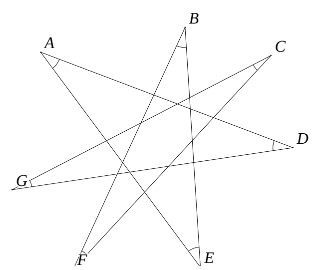
1. Let p and q be prime numbers, and let n be a whole number that satisfy:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{pq} = \frac{1}{n}.$$

   Find all possible sets of values p, q and n.

2. Find all possible sets of colored beads with the property that you can make exactly two different necklaces using all the beads in your set.

3. Prove that the sum of the measures of the angles $A+B+C+D+E+F+G$ in the figure below is $180°$.

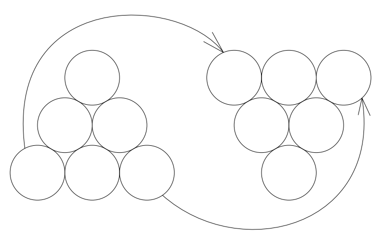


4. Let:

$$A = \{0, 1, 2, 3, \ldots, 3^k - 1\}.$$

   Show that you can choose 2^k of them such that none of the numbers you choose is equal to the arithmetic mean of two of the other numbers you chose.

5. A triangle is made of pennies with n pennies on a side and so that the tip is pointing up. What is the minimum number of pennies that must be moved to turn the triangle upside down? The figure below shows how this might be accomplished with two moves in the case $n = 3$:



6. A circular track is 100 kilometers long. Ten identical cars are arranged on it equally-spaced, so that each is 10 kilometers from the nearest neighbor cars. Each car can drive 10 kilometers on one liter of gasoline, and initially the total amount of gasoline in all the tanks is 10 liters. Show that it is possible to choose one of the cars, drive it to the next car, put all the remaining gas into one of them, then continue driving to the next and the next, moving all the gas into one of the cars each time until you have made one complete loop and wind up where you started.

   What if the cars are *not* equally-spaced at the beginning, but are arbitrarily-arranged, but still with 10 total liters of gas among them?

Solutions

1. Let p and q be prime numbers, and let n be a whole number that satisfy:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{pq} = \frac{1}{n}.$$

   Find all possible sets of values p, q and n.

   Solution: A little algebra yields:

$$n(p + q + 1) = pq.$$

   Since p and q are primes, n must be either p or q or 1. If $n = p$ or $n = q$ the equation cannot be solved, so $n = 1$, so we have:

$$p + q + 1 = pq.$$

   If both p and q are odd, then we have an even number equal to an odd number, so one of p or q must be even. The only even prime is 2, so there are two solutions: $p = 2$, $q = 3$, $n = 1$ or $p = 3$, $q = 2$, $n = 1$.

2. Find all possible sets of colored beads with the property that you can make exactly two different necklaces using all the beads in your set.

   Solution: A few observations (we'll use R for a red bead, G for a green one, Y for yellow and B for blue):

   (a) There must be at least two colors of beads, or there is only one possible necklace: a solid-colored one.

   (b) If there are exactly two colors, say R and G, then there must be more than one of each color. If there is only one R bead, for example, all necklaces look like a single R with all the rest G.

   (c) There cannot be more than three colors, since even a 4-bead necklace with one of each of the colors yields three different necklaces: $RGBY$, $GRBY$ and $GRYB$. The first is clearly different from the second, since G is between R and B in the first, and between R and Y in the second. Similarly, R changes position between the second and the third.

   (d) If a certain combination of beads yields too many necklaces, then adding beads just makes the situation worse (at least as many possibilities, and usually more).

   Thus, there are only 2 or 3 colors.

   Case 1: Exactly 2 colors. There are at least two of each color. In the easiest case, consider two of each, and this works: $RRGG$ and $RGRG$. Either the two reds are together or apart, and that forces the greens to be together or apart, respectively.

   How about two reds and three greens? Again, there are only two possibilities with the reds together or apart: $RRGGG$ or $RGRGG$.

With two reds and more than three greens, there are more than two possibilities. With four greens: $RRGGGG$, $RGRGGG$, and $RGGRGG$.

How about with three and three? No: $RRRGGG$, $RRGRGG$ and $RGRGRG$ are three different necklaces.

Case 2: Exactly 3 colors. We can't have one of each, since RGB and RBG are the same by flipping over the necklace.

If there are two or more of two colors, there are too many necklaces. Consider $RRGGB$, $RGRGB$ and $RBRGG$. Thus there are more than two of one color, and only one bead of each of the other colors. There are two situations that work:

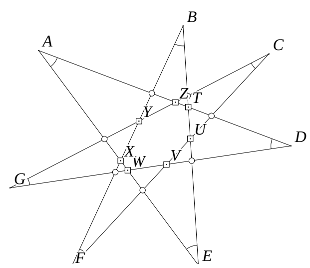
$RRGB$ and $RGRB$ (only possibilities).

$RRRGB$ and $RGRRB$ (the G and B are either together or separated and there's only one way to keep them together or to separate them).

With more than three R beads, there are too many possibilities; at least: $RRRRGB$, $RRRGRB$ and $RRGRRB$.

So there are four sets of beads that satisfy the problem: $RRGG$, $RRRGG$, $RRGB$, and $RRRGB$.

3. Prove that the sum of the measures of the angles $A+B+C+D+E+F+G$ in the figure below is $180°$.



Solution: Consider the pentagons $ATUVW$, $BUVWX$, $CVWXY$ and so on. Each has an interior angle sum of $540°$. If we add all the angles from all those pentagons together, we obtain:

$$A+B+C+D+E+F+G+4(T+U+V+W+X+Y+Z) = 7\cdot 540°.$$

But since $TUVWXYZ$ is a heptagon, its internal angles add to $900°$ and we have:

$$A+B+C+D+E+F+G+4\cdot 900° = 7\cdot 540°,$$

or

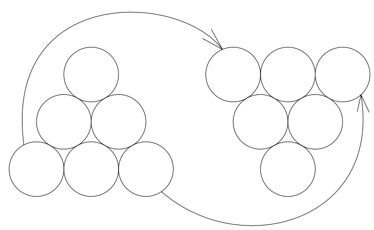
$$A+B+C+D+E+F+G = 180°.$$

We can write a similar formula for all the angles indicated with a circle on the vertex, and if we add all the similarly-derived formulas above, we obtain (writing "A" instead of "$\angle A$", et cetera) that the sum $3(A+B+C+D+E+F+G)$ is the same as the sum of the angles marked with circles. We can repeat this idea one more time to the angles marked with squares to obtain that $4(A + B + C + D + E + F + G)$ is equal to the sum of the angles marked with squares, which is the sum of the interior angles of a heptagon, which is

4. Let:
$$A = \{0, 1, 2, 3, \ldots, 3^k - 1\}.$$
Show that you can choose 2^k of them such that none of the numbers you choose is equal to the arithmetic mean of two of the other numbers you chose.

   Solution: Consider the numbers written in base 3, with leading zeros, if necessary. Choose as the subset all the numbers containing only 0's and 1's in their representation. If you choose any two different numbers from that set, they must disagree at some position in their base-3 representation. Choose the rightmost position, where one has a 0 and the other a 1. When they're added, there will be a 1 in this position and only 0's and 2's to the right. None of the rest of the numbers in the set can be half of this number, since when you double any number in the set, you'll obtain a number whose base-3 representation consists only of 0's and 2's.

5. A triangle is made of pennies with n pennies on a side and so that the tip is pointing up. What is the minimum number of pennies that must be moved to turn the triangle upside down? The figure below shows how this might be accomplished with two moves in the case $n = 3$:



   Solution: Let's use a triangle of side 4 as an illustration. Initially, from top to bottom, there are $1, 2, 3$, and 4 pennies in the rows. The inverted triangle will have $4, 3, 2$, and 1, but without some analysis, we don't know whether the top row will be in the row that initially contained $1, 2, 3$, or 4 pennies. Let's look at a couple of examples.

   In the first example, assume that the row initially containing 1 coin contains 4 at the end. In other words, there is no offset. Then the inital rows (left column) would change to contain the number on the middle column, and the difference (right column) indicates the movement into or out of the row:

1	4	$4 - 1 = 3$
2	3	$3 - 2 = 1$
3	2	$2 - 3 = -1$
4	1	$4 - 1 = -3$

   Clearly, all the negative numbers added together make the positive numbers, so with this configuration, 4 coins would have to be moved.

But there is no reason that they have to match up this way. Consider the following, where the offset is 1:

1	0	$0-1=-1$
2	4	$4-2=2$
3	3	$3-3=0$
4	2	$2-4=-2$
0	1	$1-0=1$

Three coins have to move from the lines with negative values to the lines with positive values. A total of three movements does the trick.

Here is the situation with offset 2, where again 4 coins need to be moved:

1	0	$0-1=-1$
2	0	$0-2=-2$
3	4	$4-3=1$
4	3	$3-4=-1$
0	2	$2-0=2$
0	1	$1-0=1$

With larger offsets, the number of coins increases even more.

If we examine the various offsets of the top line, we find that the best offsets (yielding a minimum number of coin moves) are, for $n = 1, 2, 3, \ldots$:

$$0, \{0,1\}, \{0,1\}, 1, \{1,2\}, \{1,2\}, 2, \{2,3\}, \{2,3\}, 3, \ldots.$$

When there are two values in set brackets, it means that either offset yields an equally small number of moves.

It is a little messy, but you can work out the formulas for the number of coins required to convert an n-penny triangle to its reverse, using an offset of k, and the formulas are, for $n + k$ odd:

$$\frac{n}{2} + \frac{n^2}{4} - k + \frac{3k^2}{4} - \frac{nk}{2} + \frac{1}{4},$$

and for $n + k$ even:

$$\frac{n}{2} + \frac{n^2}{4} - k + \frac{3k^2}{4} - \frac{nk}{2}.$$

From this information and the formulas above, we can now work out the minimum number of moves required for the first few triangle sizes, and for $n = 1, 2, 3, \ldots$ they are:

$$0, 1, 2, 3, 5, 7, 9, 12, 15, 18, 22, 26, 30, \ldots$$

In other words, one way to calculate the number, we need to start with zero moves and add the following numbers of pennies to obtain the numbers in sequence:

$$+1, +1, +1, +2, +2, +2, +3, +3, +3, +4, +4, +4, \ldots$$

If $a(n)$ is the number of moves required for a triangle formed with n pennies on a side, here is the formula for $a(n)$ which can be obtained by noticing that each sum will contain a multiple of 3 by a triangular number plus either zero, one or two additional copies of the largest number, depending on the value of n, modulo 3:

$$\begin{aligned} a(3n-1) &= n(3n+1)/2 \\ a(3n) &= 3n(n+1)/2 \\ a(3n+1) &= (n+1)(3n+2)/2 \end{aligned}$$

See: `http://www.research.att.com/ njas/sequences/A001840`

6. A circular track is 100 kilometers long. Ten identical cars are arranged on it equally-spaced, so that each is 10 kilometers from the nearest neighbor cars. Each car can drive 10 kilometers on one liter of gasoline, and initially the total amount of gasoline in all the tanks is 10 liters. Show that it is possible to choose one of the cars, drive it to the next car, put all the remaining gas into one of them, then continue driving to the next and the next, moving all the gas into one of the cars each time until you have made one complete loop and wind up where you started.

   What if the cars are *not* equally-spaced at the beginning, but are arbitrarily-arranged, but still with 10 total liters of gas among them?

   Solution: It turns out that the cars can be spaced in any way, so the first problem is just a special case of the second. Imagine that you take the initial situation and add 10 liters to the tank of one of the cars and perform the same operation, beginning with that car. Clearly, the car will make it all the way around, and at the end, will have 10 liters of gas in the tank. Now imagine the plot of the amount of gasoline in the tank of that car as it moves around the track. The minimum points of this graph will correspond to when it reaches each successive car, and all you need to do is pick the car where this graph is minimized, and this one will obviously be the correct car to choose as your starting car.