Some mathematics associated with SET

by J. B. Conrey

Here are some mathematics questions motivated by the card game SET.

(1) How many cards are there in a SET deck?
(2) If you pull three cards at random from a SET deck, what is the probability that they make a SET?

SETs can be categorized by how many of the four characteristics the three cards in the SET share. For example, the SET consisting of one plain red diamond, two plain red ovals, three plain red squiggles is a SET with two characteristics (red-ness and plain-ness) in common.

(3) If you pull three cards at random from a SET deck, what is the probability that they make a SET with three characteristics the same?
(4) If you pull three cards at random from a SET deck, what is the probability that they make a SET with two characteristics the same?
(5) If you pull three cards at random from a SET deck, what is the probability that they make a SET with one characteristic the same?
(6) If you pull three cards at random from a SET deck, what is the probability that they make an all different SET?

Two pairs of two cards each of which needs the same third card to form a set are called amicable pairs.

(7) If you pull four cards at random from a SET deck, what is the probability that they can form amicable pairs?

Three pairs of card which have the property that the three cards needed to form SETs from the pairs themselves form a SET is called a PHANTOM.

(8) If six cards can be separated into three pairs that form a PHANTOM, prove that no matter how you separate the six cards into pairs, they always form a PHANTOM.
(9) If you pull six cards at random from a SET deck, what is the probability that they make a PHANTOM?

Let $\mathbb{F}_3$ be the numbers $\{0,1,2\}$ with arithmetic mod 3. SET cards can be identified with points in the 4-dimensional space $\mathcal{S} = \mathbb{F}_3^4$. Think of each attribute – color, number, shape, and shading – as representing one dimension. $\mathcal{S}$ is a 4-dimensional *vector space*. We can add vectors in $\mathcal{S}$ just by adding the components. For example $(1,2,0,1) + (2,2,1,0) = (0,1,1,2)$, the addition is just mod 3. Also, we can do scalar multiplication in $\mathcal{S}$, for example $2(2,1,0,2) = (1,2,0,1)$. The ability to do addition and scalar multiplication is what makes it a vector space.

(10) Show that SETs are just lines in this 4-dimensional space.
(11) How many lines pass through a given point in $\mathbb{F}_3^3$? What about $\mathbb{F}_3^4$?
(12) How many planes pass through a given point in $\mathbb{F}_3^3$? What about $\mathbb{F}_3^4$?
(13) How many planes pass through a given line in $\mathbb{F}_3^3$? What about $\mathbb{F}_3^4$?

A **cap** is a collection of cards which have no SET but the addition of *any* card to the collection means that the collection has a SET.

(14) Suppose that we are playing SET with only the red ovals. What is the maximum size of a cap?
(15) Now suppose we are playing with all of the red cards. What is the maximum size of a cap?

Follow the exercises below to give two different proofs that the answer to the previous exercise is 9. First of all find a cap of size 9. Next, to show that there is no larger one, begin by assuming that there *is* a cap of size 10 in the three dimensional version of SET (i.e. using only the red cards) and we'll proceed to get a contradiction. Call this cap C. So C is supposed to be a collection of 10 points in $\mathbb{F}_3^3$ with no line. We'll show in two different ways that C can't really exist. The first way uses the pigeon-hole principle. Start by dividing $\mathbb{F}_3^3$ into three parallel planes. (Remember that we can think of $\mathbb{F}_3^3$ as the red cards. One way to split this into three parallel planes would be that the ovals are one plane, the squiggles another, and the diamonds are the third.) Call these planes H_1, H_2 and H_3.

(16) Show that the intersection of a cap in dimension 3 with any plane results in a cap of dimension 2.
(17) Let $\#(H \cap C)$ be the number of points in the intersection of the cap C with the plane H. Show that the triples $(\#(H_1 \cap C), \#(H_2 \cap C), \#(H_3 \cap C))$ are either (4,3,3) or (4,4,2).
(18) Assume that H_3 is the plane with the minimal size of intersection with C. Then there are at least 7 points of C not in H_3.
(19) Let A and B be two points in $C \cup H_3$. There are four planes that pass through A and B, one of which is H_3; call the others P_1, P_2, and P_3. One of P_1, P_2, and P_3 contains 3 points. (Why?) That plane contains 5 points of C which porves that C is not a cap. (Why?)

Now we begin the second proof. We assume again that our cap C with 10 points exists. There are lots of ways to split $\mathbb{F}_3^3$ up into three parallel hyperplanes.

(20) Show that there are 13 different ways to form a splitting into planes. (Hint: Each such splitting determines a vector perpendicular to the planes. How many such vectors are there?)

Each such splitting results in a triple (4,3,3) or (4,4,2) according to the intersections with C. Let a be the number of splittings which result in a (4,4,2) and b be the number of splittings with (4,3,3). By the previous exercise $a + b = 13$. Now we count *2-marked* planes; these are planes together with two points of C that lie in the plane.

(21) Show that there are $4\binom{10}{2} = 180$ 2-marked planes.
(22) Now show that each splitting of type (4,3,3) results in $\binom{4}{2} + \binom{3}{2} + \binom{3}{2} = 12$ different 2-marked planes and each splitting of type (4,4,2) results in $\binom{4}{2} + \binom{4}{2} + \binom{2}{2} = 13$ different 2-marked planes. Conclude that $13a + 12b = 180$. Solve the simultaneous equations in a and b and derive a contradiction.

Now do the same thing but one dimension higher to show that the maximum number of cards without a SET is 20. This time you will need to consider 2-marked and 3-marked 3-dimensional hyper-planes as well. See www.warwick.ac.uk/staff/D.Maclagan/papers/set.pdf for details.