San Jose Math Circle November 7, 2009

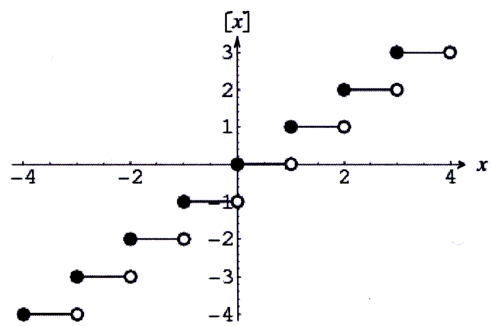
THE GREATEST INTEGER FUNCTION - THE BEGINNING

DEFINITION. The function $f : \mathbb{R} \to \mathbb{Z}$ given by $f(x) = [x]$, where $[x]$ denotes the largest integer not exceeding x, is called the *greatest integer function*.

EXAMPLES. $[2.1] = 2$, $[4.57] = 4$, $[8] = 8$, $[-2] = -2$, $[-3.4] = -4$, etc.

NOTE. The square bracket notation $[x]$ for the greatest integer function was introduced by Gauss in 1808 in his third proof of quadratic reciprocity. Some mathematicians use the notation $\lfloor x \rfloor$ and the name "floor function" to stand for the greatest integer function. This terminology has been introduced by Kenneth E. Iverson in the 1960's.

The graph of the greatest integer function is given below:



PROPERTIES OF THE GREATEST INTEGER FUNCTION:

1. $[x] = x$ if and only if x is an integer.
2. $[x] = n$ if and only if $n \leqslant x < n + 1$ if and only if $x - 1 < n \leqslant x$.
3. $[x] \leqslant x < [x] + 1$ and $x - 1 < [x] \leqslant x$.
4. $[x + n] = [x] + n$ for every x and n real numbers, with n an integer.
5. $[x] + [-x] = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ -1, & \text{if } x \notin \mathbb{Z} \end{cases}$

REMARK. The greatest integer function is very useful in applications involving data storage and data transmission.

FACT (LEGENDRE, 1808). If n is a natural number and p is a prime, then the largest power of p dividing $n!$ is denoted by $\nu_p(n)$ and is given by the formula:

$$\nu_p(n) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots$$

Note that the above sum is a finite sum, since for every given n and p eventually $\left[\frac{n}{p^j}\right] = 0$ when $p^j > n$.

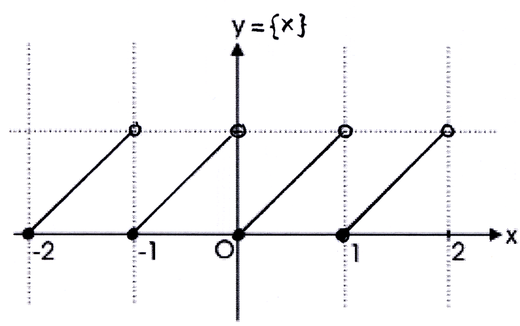
As n grows large, factorials begin acquiring tails of trailing zeros. The number of zeros is given by the exponent of 10 in the prime factorization of $n!$. Since $10 = 2 \cdot 5$ and 2 shows up more often than 5, the number of zeros is given by the exponent of 5. Thus, by Legendre's formula, the number of trailing zeros of $n!$ is given by:

$$\nu_5(n) = \left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + \left[\frac{n}{5^3}\right] + \dots$$

DEFINITION. The function $f : \mathbb{R} \to [0, 1)$ given by $f(x) = \{x\}$, where $\{x\} = x - [x]$ is called the *fractional part function* or the *sawtooth function.*

Since $\{x\} = x - [x] \Rightarrow x = [x] + \{x\}$. This is very useful in proving various other properties of the greatest integer function.

The graph of the fractional part function is below:



PROPERTIES OF THE FRACTIONAL PART FUNCTION:

1. $\{x\} = 0$ if and only if x is an integer.
2. $\{x + n\} = \{x\}$ for all real numbers x and n, with n an integer.
3. $\{x\} + \{-x\} = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \\ 1, & \text{if } x \notin \mathbb{Z} \end{cases}$
4. $0 \leq \left\{\frac{m}{n}\right\} \leq 1 - \frac{1}{|n|}$ for all integers m and n with $n \neq 0$.

1. Find the following values: $\left[\frac{7}{8}\right]$, $\left[-\frac{7}{8}\right]$, $\left[\frac{1}{2}+\left[\frac{3}{2}\right]\right]$, $\left[\frac{1}{2}\cdot\left[\frac{5}{2}\right]\right]$, $\{5.71\}$, and $\{-2.89\}$.

2. *(Mathematics Education, 1934)* Prove that: $[e]^{[\pi]}+[\pi]=[\pi]^{[e]}+[e]$ and

$$[\sqrt{2}]+[\sqrt{2}]=[\sqrt{4}] \qquad [\sqrt{3}]+[\sqrt{3}]=[\sqrt{6}] \qquad [\sqrt{8}]+[\sqrt{8}]=[\sqrt{16}]$$

3. *(2003 AMC 10 B, ♯7)* The symbolism $\lfloor x\rfloor$ denotes the largest integer not exceeding x. For example, $\lfloor 3\rfloor=3$ and $\lfloor 9/2\rfloor=4$. Compute

$$\lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+\lfloor\sqrt{3}\rfloor+\cdots+\lfloor\sqrt{16}\rfloor$$

(A) 35 (B) 38 (C) 40 (D) 42 (E) 136

4. Find $\big[[x]\big]$, $\big[\{x\}\big]$, $\{[x]\}$, and $\{\{x\}\}$. Justify your answers.

5. True or false: $[-x] = -[x]$? Justify your answer.

6. Prove or disprove: $[nx] = n[x]$ for all positive integers $n \geqslant 2$. Justify your answer.

7. Show that $[x+y] \geq [x]+[y]$ for all real numbers x and y.

8. Show that for any numbers x and y, one of the two situations can occur:

$$[x+y] = [x]+[y] \quad \text{or} \quad [x+y] = [x]+[y]+1.$$

9. *(1977 AMC 12, ♯11)* For each real number x, let $[x]$ be the largest integer not exceeding x (i.e. the integer n such that $n \leq x < n+1$). Which of the following statements are true?

I. $[x+1] = [x]+1$ for all x
II. $[x+y] = [x]+[y]$ for all x and y.
III. $[xy] = [x] \cdot [y]$ for all x and y.

(A) none (B) I only (C) I and II only (D) III only (E) all

10. Show that $[2x] = [x] + \left[x + \frac{1}{2}\right]$ for all real numbers x.

11. Prove *Hermite's identity*:

$$[nx] = [x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right]$$

for all real numbers x.

12. Show that for all real numbers x and n, with n a nonzero integer:

$$\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$$

13. Show that if m and n are integers and $n > 0$, then $\left[\frac{x+m}{n}\right] = \left[\frac{[x]+m}{n}\right]$.

14. Prove Ramanujan's Identities from the *Journal of the Indian Mathematical Society*, valid for all positive integers n:

(a) $\left[\frac{n}{3}\right] + \left[\frac{n+2}{6}\right] + \left[\frac{n+4}{6}\right] = \left[\frac{n}{2}\right] + \left[\frac{n+3}{6}\right]$

(b) $\left[\frac{1}{2} + \sqrt{n + \frac{1}{2}}\right] = \left[\frac{1}{2} + \sqrt{n + \frac{1}{4}}\right]$

(c) $\left[\sqrt{n} + \sqrt{n+1}\right] = \left[\sqrt{4n+2}\right]$

15. Calculate the sum:

$$S = \left[1+\sqrt{2}\right] + \left[\frac{2+\sqrt{3}}{2}\right] + \left[\frac{3+\sqrt{4}}{3}\right] + \cdots + \left[\frac{n+\sqrt{n+1}}{n}\right]$$

16. *(1996 Mathcounts National Team $\sharp 2$)* The greatest integer function, $[x]$, denotes the largest integer less than or equal to x. For example, $[3.5] = 3$, $[\pi] = 3$, and $[-\pi] = -4$. Find the sum of the three smallest positive solutions to $x - [x] = \frac{1}{[x]}$. Express your answer as a mixed number.

17. (a) Show that $[\sqrt{n^2+n}\,] = n$ for every natural number n.

(b) Find all natural numbers n satisfying the equation:

$$[\sqrt{1\cdot 2}\,] + [\sqrt{2\cdot 3}\,] + [\sqrt{3\cdot 4}\,] + \cdots + [\sqrt{n\cdot(n+1)}\,] = 496$$

(c) Show that

$$\log[\sqrt{1\cdot 2}] + \log[\sqrt{2\cdot 3}] + \cdots + \log[\sqrt{n(n+1)}] < n\log\frac{n+1}{2}$$

18. Solve the equation: $\left[\dfrac{x+1}{2}\right]+1=\dfrac{x-1}{3}$.

19. Solve the equation: $\left[\dfrac{3x+1}{2}\right]=x+2$.

20. *(Gazeta Matematica, Romania, 1971)* Solve the following system:

$$\begin{cases} \left[\dfrac{x+2}{3}\right] & = \dfrac{y-3}{2} \\[2ex] \left[\dfrac{y+1}{3}\right] & = \dfrac{x+3}{2} \end{cases}$$

21. Solve the following system:

$$\begin{cases} \left[\dfrac{x+1}{3}\right] & = \dfrac{3y-1}{6} \\[2ex] \left[\dfrac{y+2}{4}\right] & = \dfrac{2x-1}{3} \end{cases}$$

22. If a number ends in zeros, the zeros are called terminal zeros. For example, 520,000 has four terminal zeros, but 502,000 has just three terminal zeros. Let N equal the product of all natural numbers from 1 through 20:

$$N = 1 \times 2 \times 3 \times 4 \times \cdots \times 20$$

How many terminal zeros will N have when it is written in standard form?

23. *(2007 Mathcounts Chapter Countdown, ♯37)* Suppose that 50! is written in the form $N \times 10^x$, where N is an integer. What is the largest possible value of x?

24. *(1998 Mathcounts Chapter Team, ♯7)* In how many zeros does 75! end?

25. *(2002 Mathcounts National Target, ♯2)* In how many zeros does the decimal representation of the number 2002! end?

26. *(2002 Mathcounts National Countdown, ♯9)* In how many consecutive zeros does the product $115 \times 116 \times \cdots \times 201$ end?

27. *(2003 Mathcounts Chapter Team, ♯7)* How many zeros are at the end of (100!)(200!)(300!) when multiplied out?

28. *(2009 Stanford Mathematics Tournament)* How many consecutive zeros occur at the end of the decimal expansion of $(8!)!$?

29. *(2006 AMC 10, ♯11)* What is the tens digit in the sum $7! + 8! + 9! + \cdots + 2006!$?

    (A) 1 (B) 3 (C) 4 (D) 6 (E) 9

30. Find the sum of the last 100 digits of the number $A = 2005! + 2005$.

31. Find the exponent of 7 in the prime factorization of 90!.

32. *(2000 Mathcounts National Sprint, ♯16)* What is the largest integer value of n for which 8^n evenly divides 100!?

33. *(2003 Mathcounts State Target, ♯9)* What is the greatest positive integer n such that 3^n is a factor of 200!?

34. *(2009 Stanford Mathematics Tournament, Team Contest)* What is the largest integer n for which $\dfrac{2008!}{31^n}$ is an integer?