ARITHMETIC AND GEOMETRIC PROGRESSIONS

DEFINITION. An *arithmetic progression* is a (finite or infinite) sequence of numbers with the property that the difference between any two consecutive terms of the sequence is a constant r.

Hence a_1, a_2, …, a_n is an arithmetic progression if and only if there is a constant r such that:

$$a_2 - a_1 = r, \quad a_3 - a_2 = r, \quad \ldots, \quad a_n - a_{n-1} = r$$

or, equivalently:

$$a_2 = a_1 + r, \quad a_3 = a_2 + r, \quad \ldots, \quad a_n = a_{n-1} + r$$

EXAMPLES.

(i) 2, 5, 8, 11, 14 is an arithmetic progression with five terms and the common difference between successive terms equal to 3.

(ii) 3, 1, −1, −3, −5, −7 is an arithmetic progression with six terms and the common difference between successive terms equal to −2.

(iii) 2, 2, 2 is a constant arithmetic progression with three terms and the common difference between successive terms equal to 0.

(iv) 2, 4, 6, 8, … is an infinite arithmetic progression with the common difference between successive terms equal to 2.

TERMINOLOGY. Arithmetic progressions are sometimes called *arithmetic sequences.*

A finite arithmetic progression with terms a_1, a_2, …, a_n will be denoted by $\div\, a_1, a_2, \ldots, a_n$.

An infinite arithmetic sequence with terms a_1, a_2, … is usually denoted by (a_n).

The constant r is called the *ratio* or *common difference* of the arithmetic progression. The number a_1 is called the *first term* of the progression, a_2 is called the *2nd term* of the progression, a_3 the *3rd term*, and so on.

In example (ii) above, $r = -2$, $a_1 = 3$, and $a_3 = -1$.

A formula for the nth term of an arithmetic progression

Let a_1, a_2, $\ldots$, a_n be the first n terms of an arithmetic progression. Then:

$$\begin{aligned} a_2 &= a_1 + r \\ a_3 &= a_2 + r \\ a_4 &= a_3 + r \\ \vdots\ &\ \vdots\ \ \vdots \\ a_{n-1} &= a_{n-2} + r \\ a_n &= a_{n-1} + r \end{aligned}$$

Adding the above equations side by side we obtain:

$$a_n = a_1 + (n-1)r$$

Consequence. If $n \geqslant m$, then $a_n = a_1 + (n-1)r$ and $a_m = a_1 + (m-1)r$, so $a_n - a_m = a_1 + (n-1)r - a_1 - (m-1)r = nr - mr = (n-m)r$. Hence:

$$a_n = a_m + (n-m)r$$

A formula for the sum of the first n terms of an arithmetic progression

Let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ denote the sum of the first n terms of an arithmetic progression. Since $a_2 = a_1 + r$, $a_3 = a_1 + 2r$, $\ldots$, $a_n = a_1 + (n-1)r$, we have:

$$S_n = na_1 + [1 + 2 + \cdots + (n-1)]r = na_1 + \frac{n(n-1)}{2}r$$

Therefore:

$$S_n = \frac{n}{2}[2a_1 + (n-1)r] \quad \text{or} \quad S_n = \frac{n}{2}(a_1 + a_n)$$

Remark. If the numbers a_1, a_2, a_3, $\ldots$, a_n form an arithmetic progression, then:

$$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \cdots = 2a_1 + (n-1)r$$

1. Determine the terms a_2, a_3, a_4, a_{10} and a_{20} of an arithmetic sequence (a_n) with $a_1 = -3$ and $r = 2$.

2. Determine the 20th term of the arithmetic sequence:

$$-\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots$$

3. Find the terms a_{61} and a_{78} of an arithmetic progression (a_n) with $a_1 = 8$ and $r = 1.(3)$.

4. Consider an arithmetic sequence (a_n) with $a_1 = 5$ and $a_{45} = 16$. Is 69 a term of the sequence? Justify your answer.

5. If (a_n) is an arithmetic sequence with $a_{49} = 100$ and $r = 1.5$, then what is a_{11}?

6. If (a_n) is an arithmetic sequence with $a_3 = 13$ and $a_{34} = 137$, find a_{10}.

7. Determine the terms a_1, a_2, a_3 of the arithmetic progression:

$$a_1, a_2, a_3, 11, 15, 19, \ldots$$

8. Determine the terms a_2, a_3, a_5, a_{15}, and a_{30} of the arithmetic sequence:

$$-8, a_2, a_3, -17, a_5, \ldots$$

9. Determine the sum of the first 25 terms of an arithmetic sequence (a_n) with $a_1 = -5$ and $r = 4$.

10. Determine the sum of the first 50 terms of an arithmetic sequence (a_n) with $a_2 = -2$ and $a_7 = 28$.

11. If (a_n) is an arithmetic sequence with $a_{14} + a_{16} + a_{18} + a_{20} = 100$, find S_{33}, the sum of the first 33 terms of the progression.

12. If $S_n = 8n^2 - 6n$ represents the sum of the first n terms of an arithmetic progression (a_n), determine the first term a_1, the general term a_n, and the ratio of the progression.

13. Consider the sequence (a_n) whose general term is given by the formula:

$$a_n = \frac{6 + 11n}{4}$$

Is the sequence (a_n) an arithmetic sequence?

14. Determine an arithmetic progression $\div a_1, a_2, \ldots, a_7$ knowing that $S_7 = 7$ and $a_3 + a_6 + a_7 = 7$.

15. Determine five numbers in arithmetic progression with ratio 2 if the sum of their squares is 45. Does the problem have unique solution?

16. (2007 AMC 12 A, ♯7) Let a, b, c, d, and e be five consecutive terms in an arithmetic sequence, and suppose that $a + b + c + d + e = 30$. Which of the following can be found?

(A) a (B) b (C) c (D) d (E) e

17. (2002 AMC 10 B, ♯19) Suppose that $\{a_n\}$ is an arithmetic sequence with

$$a_1 + a_2 + \cdots + a_{100} = 100 \quad \text{and} \quad a_{101} + a_{102} + \cdots + a_{200} = 200$$

What is the value of $a_2 - a_1$?

(A) 0.0001 (B) 0.001 (C) 0.01 (D) 0.1 (E) 1

18. *(2002 AMC 12 B, ♯13)* The sum of 18 consecutive positive integers is a perfect square. What is the smallest possible value of this sum?

(A) 169 (B) 225 (C) 289 (D) 361 (E) 441

19. *(1993 AMC 12, ♯21)* Let $a_1, a_2, \ldots, a_k$ be a finite arithmetic sequence with

$$a_4 + a_7 + a_{10} = 17 \quad \text{and} \quad a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} = 77,$$

and $a_k = 13$. What is k?

(A) 16 (B) 18 (C) 20 (D) 22 (E) 24

20. (2004 AMC 10 B, ♯21) Let 1, 4, … and 9, 16, … be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

(A) 3722 (B) 3732 (C) 3914 (D) 3924 (E) 4007

21. (2003 AMC 10 B, ♯24) The first four terms in an arithmetic sequence are $x+y$, $x-y$, xy, and x/y, in that order. What is the fifth term?

(A) $-\frac{15}{8}$ (B) $-\frac{6}{5}$ (C) 0 (D) $\frac{27}{20}$ (E) $\frac{123}{40}$

22. *(1967 AMC 12, ♯35)* The roots of $64x^3 - 144x^2 + 92x - 15 = 0$ are in arithmetic progression. The difference between the largest and smallest roots is:

(A) 2 (B) 1 (C) 1/2 (D) 3/8 (E) 1/4

23. *(1969 AMC 12, ♯33)* Let S_n and T_n be the respective sums of the first n terms of two arithmetic series. If $S_n : T_n = (7n + 1) : (4n + 27)$ for all n, the ratio of the eleventh term of the first series to the eleventh term of the second series is:

(A) $4 : 3$ (B) $3 : 2$ (C) $7 : 4$ (D) $78 : 71$ (E) undetermined

24. *(1984 AIME, $\sharp$1)* Find the value of $a_2 + a_4 + a_6 + \cdots + a_{98}$ if a_1, a_2, a_3, $\ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \cdots + a_{98} = 137$.

25. (1999 New Mexico Mathematics Contest, First Round, $\sharp$5) The positive numbers $a_1 = 4$, $a_2 = 7$, $a_3 = 10$, $\ldots$ form an arithmetic progression, that is: $a_n = a_{n-1} + r$ where r is fixed and $n = 1, 2, 3, \ldots$. Calculate:

    (a) a_{33}

    (b) $\dfrac{1}{\sqrt{a_1} + \sqrt{a_2}} + \dfrac{1}{\sqrt{a_2} + \sqrt{a_3}} + \cdots + \dfrac{1}{\sqrt{a_{32}} + \sqrt{a_{33}}}$

26. *(2005 AIME 1, ♯2)* For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence $1, 4, 7, \ldots$. For how many values of k does S_k contain the term 2005?

27. *(2003 AIME 2, ♯8)* Find the eighth term of the sequence $1440, 1716, 1848, \ldots$, whose terms are formed by multiplying the corresponding terms of two arithmetic sequences.

28. Show that $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ cannot be terms of an arithmetic progression.

29. Let a, b, and c be relatively prime positive integers. Show that if $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$ are terms of the same arithmetic sequence, then a, b, and c must be perfect squares.

30. *(1973 USAMO, ♯5)* Show that the cube roots of three distinct prime numbers cannot be the three terms (not necessarily consecutive) of an arithmetic progression.

31. *(1980 USAMO, ♯2)* Determine the maximum number of different three-term arithmetic progressions which can be chosen from a sequence of n real numbers

$$a_1 < a_2 < \cdots < a_n.$$

32. *(1991 IMO, ♯2)* Let $n > 6$ be an integer and let $a_1, a_2, \ldots, a_k$ be all the positive integers less than n and relatively prime to n. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n must either be a prime number or a power of 2.

DEFINITION. A *geometric progression* is a (finite or infinite) sequence of numbers with the property that the quotient of any two consecutive terms of the sequence is a constant q.

Hence $b_1, b_2, \ldots, b_n$ is a geometric progression if and only if there is a constant q such that:

$$\frac{b_2}{b_1} = q, \quad \frac{b_3}{b_2} = q, \quad \ldots, \quad \frac{b_n}{b_{n-1}} = q$$

or, equivalently:

$$b_2 = qb_1, \quad b_3 = qb_2, \quad \ldots, \quad b_n = qb_{n-1}$$

EXAMPLES.

(i) 2, 10, 50, 250, 1250 is a geometric progression with five terms and the quotient between successive terms equal to 5.

(ii) 3, −6, 12, −24, 48, −96 is a geometric progression with six terms and the quotient between successive terms equal to −2.

(iii) 7, 7, 7 is a constant geometric progression with three terms and the quotient between successive terms equal to 1.

(iv) 2, 4, 8, 16, … is an infinite geometric progression with the quotient between successive terms equal to 2.

TERMINOLOGY. Geometric progressions are sometimes called *geometric sequences.*

A finite geometric progression with terms $b_1, b_2, \ldots, b_n$ will be denoted by $\odot\, b_1, b_2, \ldots, b_n$.

An infinite geometric sequence with terms $b_1, b_2, \ldots$ is usually denoted by (b_n).

The constant q is called the *ratio* of the geometric progression. The number b_1 is called the *first term* of the progression, b_2 is called the *2nd term* of the progression, b_3 the *3rd term*, and so on.

In example (ii) above, $q = -2$, $b_1 = 3$, and $b_4 = -24$.

A formula for the nth term of a geometric progression

Let b_1, b_2, …, b_n be the first n terms of a geometric progression in which $q \neq 0$, $q \neq 1$ and $b_1 \neq 0$. Then:

$$\begin{aligned} b_2 &= qb_1 \\ b_3 &= qb_2 \\ b_4 &= qb_3 \\ \vdots\ & \vdots\ \ \vdots \\ b_{n-1} &= qb_{n-2} \\ b_n &= qb_{n-1} \end{aligned}$$

Note that the assumptions imply that no term of the progression is equal to zero. Multiplying the above equations side by side we obtain:

$$b_n = b_1 q^{n-1}$$

Consequence. With the above assumptions, if $n \geqslant m$, then $b_n = b_1 q^{n-1}$ and $b_m = b_1 q^{m-1}$, so $\dfrac{b_n}{b_m} = \dfrac{b_1 q^{n-1}}{b_1 q^{m-1}} = q^{n-m}$. Hence:

$$b_n = b_m q^{n-m}$$

A formula for the sum of the first n terms of a geometric progression

Let $S_n = b_1 + b_2 + b_3 + \cdots + b_n$ denote the sum of the first n terms of a geometric progression with $q \neq 1$, $q \neq 0$, and $b_1 \neq 0$.
Since $b_2 = b_1 q$, $b_3 = b_1 q^2$, …, $b_n = b_1 q^{n-1}$, we have:

$$S_n = b_1 + b_1 q + b_1 q^2 + \cdots + b_1 q^{n-1} = b_1\left(1 + q + q^2 + \cdots + q^{n-1}\right)$$

But

$$1 + q + q^2 + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}$$

Consequently,

$$S_n = b_1 \frac{q^n - 1}{q - 1}$$

Remark. If the numbers b_1, b_2, b_3, …, b_n form a geometric progression, then:

$$b_1 b_n = b_2 b_{n-1} = b_3 b_{n-2} = \cdots = b_1^2 q^{n-1}$$

The nature and behavior of a geometric sequence depends on the value of q and the sign of b_1.

(a) If $q > 1$ and $b_1 > 0$, then all terms are positive and increasing. If $q > 1$ and $b_1 < 0$, then all terms are negative and decreasing.

(b) If $0 < q < 1$ and $b_1 > 0$, then all terms are positive and decreasing. If $0 < q < 1$ and $b_1 < 0$, then all terms are negative and increasing.

(c) If $q < 0$, then the terms alternate in sign.

(d) If $q = 0$, then all terms of the progression are 0, except possibly for b_1.

(e) If $q = 1$, then all terms of the progression are equal to b_1.

33. Determine the nth term of each of the following geometric sequences:

(a) $-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$

(b) $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, \ldots$

(c) $-32, 16, -8, \ldots$

(d) $\frac{1}{3}, \frac{\sqrt{3}}{3}, 1, \ldots$

34. Determine the terms b_1, b_4, b_8, and b_{15} of each of the following geometric progressions:

(a) b_1, -8, 16, b_4, $\ldots$

(b) b_1, 12, 6, b_4, $\ldots$

(c) b_1, 1, $\frac{1}{3}$, b_4, $\ldots$

35. Determine the first two terms of a geometric progression satisfying the following conditions:

    (a) $b_8 = 27b_2$ and $b_1 + b_2 + b_3 = 4 + \sqrt{3}$

    (b) $b_6 = \dfrac{4}{9}$ and $b_7 = -\dfrac{4}{27}$

    (c) $b_{11} = 6144$ and $b_{15} = 32b_{10}$

(d) $b_4 = -\frac{2}{9}$ and $q = -\frac{1}{3}$

(e) $b_5 = 54$ and $b_7 = 486$

36. Is the sequence $(b_n)_{n \geqslant 1}$ with $b_n = \frac{3^{n+1}}{4^n}$ a geometric sequence? If so, what is its ratio?

37. Consider the sequence (v_n) with general term $v_n = 2(\sqrt{3})^n$ for all $n \geqslant 1$.

(a) Establish whether or not (v_n) is a geometric progression.

(b) Which of the following numbers are terms of the sequence (v_n): 18, 36, $162\sqrt{3}$, -729, 1458?

38. If (b_n) is a geometric progression with $b_1 = \frac{4}{5}$ and $b_4 = \frac{64}{48}$, find b_8 and S_{30}.

39. Determine the geometric progression (b_n) satisfying the following conditions:

$$\begin{cases} b_5 - b_1 = -1218 \\ b_4 - b_2 = -420 \end{cases}$$

40. Calculate the following sums:

(a) $1 + \dfrac{1}{5} + \dfrac{1}{5^2} + \cdots + \dfrac{1}{5^{14}}$

(b) $\dfrac{1}{4} - \dfrac{1}{4^2} + \dfrac{1}{4^3} - \dfrac{1}{4^4} + \cdots + \dfrac{1}{4^{15}}$

(c) $1 + 11 + 111 + \cdots + \underbrace{111\ldots11}_{n \quad \text{times}}$

41. *(1950 AMC 12, ♯5)* If five geometric means are inserted between 8 and 5832, the fifth term in the geometric series is:

(A) 648 (B) 832 (C) 1168 (D) 1944 (E) none of these

42. *(1953 AMC 12, ♯25)* In a geometric progression whose terms are positive, any term is equal to the sum of the next two following terms. Then the common ratio is:

(A) 1 (B) about $\frac{\sqrt{5}}{2}$ (C) $\frac{\sqrt{5}-1}{2}$ (D) $\frac{1-\sqrt{5}}{2}$ (E) $\frac{2}{\sqrt{5}}$

43. *(1955 AMC 12, ♯45)* Given a geometric sequence with the first term $\neq 0$ and $r \neq 0$ and an arithmetic sequence with the first term $= 0$, a third sequence $1, 1, 2, \ldots$ is formed by adding the corresponding terms of the two given sequences. The sum of the first ten terms of the third sequence is:

(A) 978 (B) 557 (C) 467 (D) 1068 (E) not possible to determine from the information given

44. *(1959 AMC 12, ♯12)* By adding the same constant to each of 20, 50, 100 a geometric progression results. The common ratio is:

(A) $\frac{5}{3}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{1}{2}$ (E) $\frac{1}{3}$

45. *(1959 AMC 12, ♯39)* Let S be the sum of the first nine terms of the sequence

$$x + a,\ x^2 + 2a,\ x^3 + 3a, \ldots$$

Then S equals:

(A) $\dfrac{50a + x + x^8}{x + 1}$ (B) $50a - \dfrac{x + x^{10}}{x - 1}$ (C) $\dfrac{x^9 - 1}{x + 1} + 45a$ (D) $\dfrac{x^{10} - x}{x - 1} + 45a$
(E) $\dfrac{x^{11} - x}{x - 1} + 45a$

46. *(1967 AMC 12, ♯36)* Given a geometric progression of five terms, each a positive integer less that 100, the sum of the five terms is 211. If S is the sum of those in the progression which are squares of integers, then S is:

(A) 0 (B) 91 (C) 133 (D) 195 (E) 211

REMARK 1. Let (b_n) be a geometric sequence with common ratio q and let P_n denote the product of the first n terms of the sequence.

$$P_n = b_1 b_2 b_3 \cdot \dots \cdot b_n$$

Since $b_2 = b_1 q$, $b_3 = b_1 q^2$, $\dots$, $b_n = b_1 q^{n-1}$ we obtain:

$$P_n = b_1 \cdot b_1 q \cdot b_1 q^2 \cdot \dots \cdot b_1 q^{n-1} = b_1^n q^{1+2+\dots+(n-1)} = b_1^n q^{n(n-1)/2}$$

Note that $b_1^n q^{n(n-1)/2} = b_1^{n/2} b_1^{n/2} q^{n(n-1)/2} = b_1^{n/2} (b_1 q^{n-1})^{n/2} = b_1^{n/2} b_n^{n/2} = (b_1 b_n)^{n/2}$. Therefore:

$$P_n = (b_1 b_n)^{n/2}$$

REMARK 2. Let (b_n) be a geometric sequence with common ratio $q \neq 1$ such that every term of the sequence is nonzero. Consider the sequence formed by taking the reciprocal of every term in the progression and note that the following property holds:

$$\frac{1/b_{n+1}}{1/b_n} = \frac{b_n}{b_{n+1}} = \frac{b_n}{b_n q} = \frac{1}{q}$$

Hence $(1/b_n)$ is a geometric sequence with ratio $1/q$.

Let S_n be the sum of the first n terms of (b_n) and let $\mathfrak{S}_\mathfrak{n}$ be the sum of the first n terms of $(1/b_n)$. Then:

$$\mathfrak{S}_\mathfrak{n} = \frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_n} = \frac{1}{b_1} \cdot \frac{(1/q)^n - 1}{1/q - 1} = \frac{1}{b_1} \cdot \frac{\dfrac{1}{q^n} - 1}{\dfrac{1}{q} - 1} = \frac{1}{b_1 q^{n-1}} \cdot \frac{1 - q^n}{1 - q}$$

Thus

$$\mathfrak{S}_\mathfrak{n} = \frac{1}{b_n} \cdot \frac{q^n - 1}{q - 1}$$

Since $S_n = b_1 \cdot \dfrac{q^n - 1}{q - 1}$, it follows that $\dfrac{\mathfrak{S}_\mathfrak{n}}{S_n} = \dfrac{1}{b_1 b_n}$, and consequently:

$$\mathfrak{S}_\mathfrak{n} = \frac{S_n}{b_1 b_n}$$

REMARK 3. Let (b_n) be a geometric sequence with common ratio $q \neq 1$. Let p be a positive integer such that $q^p \neq 1$. Consider the sequence obtained by raising the terms of the progression (b_n) to the pth power and note that:

$$\frac{b_{n+1}^p}{b_n^p} = \left(\frac{b_{n+1}}{b_n}\right)^p = q^p$$

This shows that (b_n^p) is a geometric sequence with ratio q^p.

Let Σ_n^p be the sum of the first n terms of (b_n^p). We have:

$$\Sigma_n^p = b_1^p + b_2^p + \cdots + b_n^p = b_1^p \cdot \frac{(q^p)^n - 1}{q^p - 1}$$

We have obtained the following formula for the sum of the pth powers of the first n terms of a geometric progression:

$$\Sigma_n^p = b_1^p \cdot \frac{q^{np} - 1}{q^p - 1}$$

In particular, the sum of the squares of the first n terms of a geometric progression (b_n) with ratio $q \neq \pm 1$ is

$$\Sigma_n^2 = b_1^2 + b_2^2 + \cdots + b_n^2 = b_1^2 \cdot \frac{q^{2n} - 1}{q^2 - 1}$$

47. *(1971 AMC 12, ♯29)* Given the progression $10^{1/11}$, $10^{2/11}$, $10^{3/11}$, $10^{4/11}$, $\ldots$, $10^{n/11}$, the least positive integer n such that the product of the first n terms of the progression exceeds 100,000 is:

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

48. *(1971 AMC 12, ♯33)* If P is the product of n quantities in geometric progression, S is their sum, and S' the sum of their reciprocals, then P in terms of S, S', and n is:

(A) $(SS')^{n/2}$ (B) $(S/S')^{n/2}$ (C) $(SS')^{n-2}$ (D) $(S/S')^{n}$ (E) $(S'/S)^{(n-1)/2}$

49. *(1972 AMC 12, ♯16)* There are two positive numbers that may be inserted between 3 and 9 such that the first three are in geometric progression, while the last three are in arithmetic progression. The sum of these two positive numbers is:

(A) $13\frac{1}{2}$ (B) $11\frac{1}{4}$ (C) $10\frac{1}{2}$ (D) 10 (E) $9\frac{1}{2}$

50. *(1972 AMC 12, ♯19)* The sum of the first n terms of the sequence

$$1,(1+2),(1+2+2^2),\ldots,(1+2+2^2+\cdots+2^{n-1})$$

in terms of n is:

(A) 2^n (B) 2^n-n (C) $2^{n+1}-n$ (D) $2^{n+1}-n-2$ (E) $n\cdot 2^n$

51. *(1974 AMC 12, ♯21)* In a geometric series of positive terms the difference between the fifth and fourth terms is 576, and the difference between the second and first terms is 9. What is the sum of the first five terms of this series?

(A) 1061 (B) 1023 (C) 1024 (D) 768 (E) none of these

52. *(1976 AMC 12, ♯4)* Let a geometric progression with n terms have first term 1, common ratio r and sum s, where r and s are not zero. The sum of the geometric progression formed by replacing each term of the original progression by its reciprocal is:

(A) $\frac{1}{s}$ (B) $\frac{1}{r^n s}$ (C) $\frac{s}{r^{n-1}}$ (D) $\frac{r^n}{s}$ (E) $\frac{r^{n-1}}{s}$

53. *(1977 AMC 12, ♯13)* If a_1, a_2, a_3, $\ldots$ is a sequence of positive numbers such that $a_{n+2} = a_n a_{n+1}$ for all positive integers n, then the sequence a_1, a_2, a_3, $\ldots$ is a geometric progression:

(A) for all positive values of a_1 and a_2 (B) if and only if $a_1 = a_2$
(C) if and only if $a_1 = 1$ (D) if and only if $a_2 = 1$ (E) if and only if $a_1 = a_2 = 1$

54. *(1981 AMC 12, ♯14)* In a geometric sequence of real numbers, the sum of the first two terms is 7, and the sum of the first six terms is 91. The sum of the first four terms is

(A) 28 (B) 32 (C) 35 (D) 49 (E) 84

55. *(1994 AMC 12, ♯20)* Suppose x, y, z is a geometric sequence with common ratio r and $x \neq y$. If x, $2y$, $3z$ is an arithmetic sequence, then r is:

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 2 (E) 4

56. *(2002 AMC 12 B, ♯9)* If a, b, c, d are positive real numbers such that a, b, c, d form an increasing arithmetic sequence and a, b, d form a geometric sequence, then $\frac{a}{d}$ is:

(A) $\frac{1}{12}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

57. *(2003 AMC 10 B, ♯8 and 2003 AMC 12 B, ♯6)* The second and fourth terms of a geometric sequence are 2 and 6. Which of the following is a possible first term?

(A) $-\sqrt{3}$ (B) $-\dfrac{2\sqrt{3}}{3}$ (C) $-\dfrac{\sqrt{3}}{3}$ (D) $\sqrt{3}$ (E) 3

58. *(2004 AMC 10 A, ♯18 and 2004 AMC 12 A, ♯14)* A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression?

(A) 1 (B) 4 (C) 36 (D) 49 (E) 81

59. *(2003 AIME I, ♯8)* In an increasing sequence of four positive integers, the first three terms form an arithmetic progression, the last three terms form a geometric progression, and the first and fourth terms differ by 30. Find the sum of the four terms.

60. Show that if the nonzero real numbers a_1, a_2, $\ldots$, a_n, $n \geq 3$ satisfy the equality:

$$(a_1^2 + a_2^2 + \cdots + a_{n-1}^2)(a_2^2 + a_3^2 + \cdots + a_n^2) = (a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n)^2,$$

then they form a geometric progression.

61. *(Grigore Moisil Math Competition, Romania, 2007)*

(a) Find three real numbers $x < y < z$ in geometric progression satisfying the following conditions:

$$x + y + z = \frac{19}{18}$$
$$x^2 + y^2 + z^2 = \frac{133}{324}$$

(b) Find positive rational numbers a, b, c, r given that $a < b < c$ form an arithmetic progression with ratio $r + 1$ and $a < b + 2 < c + 12$ form a geometric progression with ratio $r + a$.

62. *(2007 Romanian Math Olympiad, Round I)*

(a) Show that a sequence $(a_n)_{n\geq 1}$ of positive real numbers is a arithmetic progression if and only if

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\cdots+\frac{1}{a_{n-1}a_n}=\frac{n-1}{a_1a_n} \quad \text{for all} \quad n\geq 2$$

(b) Let a and b be two real numbers. Find three numbers x, y, and z such that x, y, z form a geometric progression, x, $y+a$, z form an arithmetic progression, and x, $y+a$, $z+b$ form a geometric progression.

63. *(Adolf Haimovici National Contest, Romania, 2007)* If $(a_n)_{n\geq 1}$ is a geometric progression with positive terms satisfying:

$$a_1 + a_2 + \cdots + a_{2007} = 2 \quad \text{and} \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2007}} = 1,$$

find the product $a_1 a_2 \ldots a_{2007}$.

64. *(Gheorghe Lazar Math Competition, Romania, 2007)* Let $r > 0$ and let $(a_n)_{n\geq 1}$ be a sequence with positive terms. Show that $(a_n)_{n\geq 1}$ is a geometric sequence with ratio r if and only if the following equality holds:

$$\sqrt{\frac{1}{r^2} + \frac{1}{a_1^2} + \frac{1}{a_2^2}} + \sqrt{\frac{1}{r^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2}} + \cdots + \sqrt{\frac{1}{r^2} + \frac{1}{a_{n-1}^2} + \frac{1}{a_n^2}} = \frac{n-1}{r} + \frac{1}{a_1} - \frac{1}{a_n}$$

for all $n \geq 2$.

65. Show that if an arithmetic progression of natural numbers contains a perfect square, then it contains infinitely many perfect squares.

66. Let $(a_n)_{n\geq 1}$ be an arithmetic sequence and let $(b_n)_{n\geq 1}$ be a geometric sequence. Calculate the sum:

$$S = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}$$

67. *(2002 AIME 2, ♯3)* It is given that $\log_6 a + \log_6 b + \log_6 c = 6$, where a, b, and c are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a + b + c$.

68. *(2005 AIME 2, ♯3)* An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has sum 10 times the sum of the original series. The common ratio of the original series is m/n, where m and n are relatively prime positive integers. Find $m + n$.

69. *(2006 AIME 1, ♯9)* The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

70. *(2004 AIME 2, ♯9)* A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth are in arithmetic progression, and, in general, for all $n \geq 1$, the terms a_{2n-1}, a_{2n}, and a_{2n+1} are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be the greatest term in this sequence that is less than 1000. Find $n + a_n$.

71. *(2007 AIME 2, ♯12)* The increasing geometric sequence x_0, x_1, x_2, ... consists entirely of integral powers of 3. Given that

$$\sum_{n=0}^{7} \log_3(x_n) = 308 \quad \text{and} \quad 56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \geq 57,$$

find $\log_3(x_{14})$.

72. *(1999 AIME, ♯1)* Find the smallest prime that is the fifth term of an increasing arithmetic sequence, all four preceding terms also being prime.

73. *(1989 AIME, ♯7)* If the integer k is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic series. Find k.

74. *(1982 AMC 12, ♯8)* By definition $r! = r(r-1)\cdots 1$ and

$$\binom{j}{k} = \frac{j!}{k!(j-k)!},$$

where r, j, k are positive integers and $k < j$. If $\binom{n}{1}$, $\binom{n}{2}$, $\binom{n}{3}$ form an arithmetic progression with $n > 3$, then n equals:

(A) 5 (B) 7 (C) 9 (D) 11 (E) 12