Close Encounters of the Stirling Kind

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Acknowledgement: Almost everything I know about Stirling numbers comes from either Graham/Knuth/Patashnik *Concrete Math* or from Benjamin/Quinn, *Proofs that Really Count*. So most of what you see below is from one or the other of those sources.

Warm-up 1. Simplify each of the following:

x

$x\cdot(x+1)$

$x\cdot(x+1)\cdot(x+2)$

$x\cdot(x+1)\cdot(x+2)\cdot(x+3)$

What is the pattern?

Can you explain how to compute, say, the coefficient of x^3 in the seventh item in this list?

Warm-up 2. Compute the values of h_n below, and explain a pattern for computing them.

$$\frac{1}{1}=\frac{h_1}{1!}$$

$$\frac{1}{1}+\frac{1}{2}=\frac{h_2}{2!}$$

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}=\frac{h_3}{3!}$$

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{h_4}{4!}$$

Warm-up 3. What do the two problems above have to do with each other, and with the topic for this talk?

James Stirling (1692-1770) wrote first about what are now called Stirling numbers of the second kind, and then second about what are now called Stirling numbers of the first kind.

Stirling numbers of the second kind are noted $S(n, k)$ or sometimes $S_n^{(k)}$ but as usual I favor Knuth's notation and pronunciation "*n* subset *k*" (as do Benjamin and Quinn): $\left\{ n \atop k \right\}$.

But what does it mean? It means the number of ways to partition a set of n things (usually the integers 1 through n) into k nonempty subsets. There are non-Stirling variations, like what if the things are identical? Or what if the order of the sets matters? But with Stirling, the sets are considered interchangeable but the elements are not.

I like to call them "Stirling subset numbers" because otherwise I can never remember which ones come second.

Problem 1. Make "Stirling's second triangle" by computing the first several values.

Problem 2. Find useful formulas for some of the columns or diagonals of Stirling subset triangle.

Problem 3. Find a useful recursive formula for computing Stirling subset numbers.

Problem 4. Prove that x^n = the sum of $\left\{ n \atop k \right\}$ $x^{\underline{k}}$. (Do you know that notation with the underlined exponent for the "falling factorial"? Or have you seen it as $x_{(k)}$ perhaps?)

Stirling numbers of the first kind, sometimes called Stirling cycle numbers, are similar to those of the second kind, only now we think of the elements of each subset being written in order around a circle. As before, Knuth encourages us to say "*n* cycle *k*" and to write $\left[n \atop k \right]$.

Problem 5. Make "Stirling's first triangle" by computing the first several values.

Problem 6. Explain why the sum of $\left[n \atop k \right]$ is n!

Problem 7. What columns or diagonals are convenient to compute?

Problem 8. Write a recursive formula that makes it reasonably easy to fill in the triangle.

Problem 9. What is now the obvious fact about odd and even k, and why is it true? [Hint: split according to whether n is alone.]

Problem 10. Prove that $x^{\overline{n}} = \sum_{k=1}^{n} \left[n \atop k \right]$. Use two methods: (1) prove that both sides satisfy the recursive formula and the initial conditions, and (2) compare, say, the coefficient of x^3 on both sides by looking at the smallest number of each of three cycles and then placing the rest.

Problem 11. Relate the numerators of H_n (don't simplify fractions!) to Stirling cycle numbers. Hint for method 1: look at the number of elements in the second cycle. Or 2: look at the smallest number in the second cycle.