San Jose Math Circle September 30, 2009

PROBLEM CHALLENGE
by Lucian Sega

1. Of the following ten integers, only one is a perfect square. Which one? (DO NOT USE A CALCULATOR)

   (a) 8344572651 (b) 7955896032 (c) 1695032253 (d) 4906358264 (e) 1782570645

   (f) 5729581636 (g) 3213046377 (h) 2032918848 (i) 2973562479 (j) 3567659100

2. Let A and B be positive integers that can be written each as a sum of two perfect squares. Show that AB can also be written as a sum of two perfect squares.

3. Let N be a 100-digit number whose digits are, all but one, equal to 5. Can N be a perfect square?

4. (a) Find four positive integers a, b, c, and d such that $a^2 + b^2$, $a^2 + b^2 + c^2$, and $a^2 + b^2 + c^2 + d^2$ are all perfect squares.
   (b) Does there exist a sequence of perfect squares with the property that the sum of the first n terms of the sequence is also a perfect square, for all $n \geq 1$?

5. For a positive integer n, the nth triangular number t_n is defined to be:

$$t_n = 1 + 2 + 3 + \cdots + n$$

   In July 1796, Gauss proved that every positive integer is the sum of three or fewer triangular numbers. Prove that Gauss's discovery implies that every positive integer of the form $8k + 3$ can be expressed as a sum of three odd squares.

6. Does there exist a positive integer $n > 1$ such that $1^2 + 2^2 + 3^2 + \cdots + n^2$ is a perfect square?

7. An even natural number n is called a square dance number if the numbers from 1 to n can be paired in such a way that the sum of each pair is a perfect square.
   (a) Show that 48 is a square dance number.
   (b) Find all square dance numbers.