PERFECT SQUARES

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1. Let $s_n = n^2$ be the nth perfect square, where $n \geq 1$ is a positive integer. Show that s_n exceeds its predecessor s_{n-1} be the sum of the two roots; that is, show that:

$$s_n = s_{n-1} + \sqrt{s_n} + \sqrt{s_{n-1}}$$

[This appears in 1225 in Fibonacci's *Liber Quadratorum (The Book of Squares)*]

2. Show that the nth perfect square is the sum of the first n odd numbers; that is:

$$n^2 = 1 + 3 + 5 + \cdots + (2n - 1)$$

[This was known to Pythagoreans in the sixth century BC and first appears in Europe in Fibonacci's *Liber Quadratorum*]

3. For a positive integer n, the nth triangular number t_n is defined to be:

$$t_n = 1 + 2 + 3 + \cdots + n$$

Show that 8 times a triangular number plus 1 is a perfect square; that is, show that:

$$8t_n + 1 = s_{2n+1}$$

[This appears in the second century in Plutarch's *Platonic Questions*]

4. Prove that the sum of two consecutive triangular numbers is always a perfect square.

5. Of the following ten integers, only one is a perfect square. Which one? (DO NOT USE A CALCULATOR)

(a) 8344572651 (b) 7955896032 (c) 1695032253 (d) 4906358264 (e) 1782570645

(f) 5729581636 (g) 3213046377 (h) 2032918848 (i) 2973562479 (j) 3567659100

[New Mexico Mathematics Contest, 1990]

6. Let A and B be positive integers that can be written each as a sum of two perfect squares. Show that AB can also be written as a sum of two perfect squares.

7. Let N be a 100-digit number whose digits are, all but one, equal to 5. Can N be a perfect square?

8. (a) Find four positive integers a, b, c, and d such that $a^2 + b^2$, $a^2 + b^2 + c^2$, and $a^2 + b^2 + c^2 + d^2$ are all perfect squares.

   (b) Does there exist a sequence of perfect squares with the property that the sum of the first n terms of the sequence is also a perfect square, for all $n \geq 1$?

9. In July 1796, Gauss proved that every positive integer is the sum of three or fewer triangular numbers. Prove that Gauss's discovery implies that every positive integer of the form $8k + 3$ can be expressed as a sum of three odd squares.

10. Prove that:

$$1^2 + 2^2 + 3^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

11. Prove the following formulas due to Mariares (1913):

$$1^2 + 3^2 + 5^2 + \cdots + n^2 = \binom{n+2}{3} \quad \text{if n is odd}$$

$$2^2 + 4^2 + 6^2 + \cdots + n^2 = \binom{n+2}{3} \quad \text{if n is even}$$

12. Does there exist a positive integer $n > 1$ such that $1^2 + 2^2 + 3^2 + \cdots + n^2$ is a perfect square? [*Ladies' Diary*, 1792]

    This problem was posed by Edouard Lucas in 1875 in Annales de Mathematique Nouvelles. It was solved by G.N. Watson in 1918.

13. *(Diophantus' Problem)* Find three integers in arithmetic progression given that the sum of any two of them is a perfect square.

14. Show that for all positive integers n, the sum:

$$1^3 + 2^3 + 3^3 + \cdots + n^3$$

    is a perfect square.

15. Show that the cube of every positive integer can be written as a difference of two perfect squares.

16. An even natural number n is called a square dance number if the numbers from 1 to n can be paired in such a way that the sum of each pair is a perfect square.

   (a) Show that 48 is a square dance number.

   (b) Find all square dance numbers.