PERFECT SQUARES
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1. Let \( s_n = n^2 \) be the \( n \)th perfect square, where \( n \geq 1 \) is a positive integer. Show that \( s_n \) exceeds its predecessor \( s_{n-1} \) be the sum of the two roots; that is, show that:

\[
\begin{align*}
  s_n &= s_{n-1} + \sqrt{s_n} + \sqrt{s_{n-1}} \\
[\text{This appears in 1225 in Fibonacci’s } \textit{Liber Quadratorum (The Book of Squares)}] 
\end{align*}
\]

2. Show that the \( n \)th perfect square is the sum of the first \( n \) odd numbers; that is:

\[
  n^2 = 1 + 3 + 5 + \cdots + (2n - 1) 
\]

[This was known to Pythagoreans in the sixth century BC and first appears in Europe in Fibonacci’s \textit{Liber Quadratorum}\]

3. For a positive integer \( n \), the \( n \)th triangular number \( t_n \) is defined to be:

\[
  t_n = 1 + 2 + 3 + \cdots + n 
\]

Show that 8 times a triangular number plus 1 is a perfect square; that is, show that:

\[
  8t_n + 1 = s_{2n+1} 
\]

[This appears in the second century in Plutarch’s \textit{Platonic Questions}\]

4. Prove that the sum of two consecutive triangular numbers is always a perfect square.

5. Of the following ten integers, only one is a perfect square. Which one? (DO NOT USE A CALCULATOR)

(a) 8344572651   (b) 7955896032   (c) 1695032253   (d) 4906358264   (e) 1782570645
(f) 5729581636   (g) 3213046377   (h) 2032918848   (i) 2973562479   (j) 3567659100
[New Mexico Mathematics Contest, 1990]

6. Let \( A \) and \( B \) be positive integers that can be written each as a sum of two perfect squares. Show that \( AB \) can also be written as a sum of two perfect squares.
7. Let $N$ be a 100-digit number whose digits are, all but one, equal to 5. Can $N$ be a perfect square?

8. (a) Find four positive integers $a$, $b$, $c$, and $d$ such that $a^2 + b^2$, $a^2 + b^2 + c^2$, and $a^2 + b^2 + c^2 + d^2$ are all perfect squares.

(b) Does there exist a sequence of perfect squares with the property that the sum of the first $n$ terms of the sequence is also a perfect square, for all $n \geq 1$?

9. In July 1796, Gauss proved that every positive integer is the sum of three or fewer triangular numbers. Prove that Gauss’s discovery implies that every positive integer of the form $8k + 3$ can be expressed as a sum of three odd squares.

10. Prove that:

\[ 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \]

11. Prove the following formulas due to Mariareas (1913):

\[ 1^2 + 3^2 + 5^2 + \cdots + n^2 = \binom{n+2}{3} \text{ if } n \text{ is odd} \]
\[ 2^2 + 4^2 + 6^2 + \cdots + n^2 = \binom{n+2}{3} \text{ if } n \text{ is even} \]

12. Does there exist a positive integer $n > 1$ such that $1^2 + 2^2 + 3^2 + \cdots + n^2$ is a perfect square? [Ladies’ Diary, 1792]

This problem was posed by Edouard Lucas in 1875 in Annales de Mathematique Nouvelles. It was solved by G.N. Watson in 1918.

13. (Diophantus’ Problem) Find three integers in arithmetic progression given that the sum of any two of them is a perfect square.

14. Show that for all positive integers $n$, the sum:

\[ 1^3 + 2^3 + 3^3 + \cdots + n^3 \]

is a perfect square.

15. Show that the cube of every positive integer can be written as a difference of two perfect squares.
16. An even natural number $n$ is called a square dance number if the numbers from 1 to $n$ can be paired in such a way that the sum of each pair is a perfect square.

(a) Show that 48 is a square dance number.
(b) Find all square dance numbers.