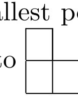
San Jose Math Circle 2009/2010

Counting With Bijections - Ivan Matić

1. Given an n-element set S, in how many ways can we choose a subset X of S?
2. Given an n-element set S, in how many ways can we chose an ordered pair (X, Y) of subsets of S?
3. Given an n-element set S, in how many ways can we choose an ordered pair (X, Y) of subsets of S such that $X \cup Y = S$?
4. Given an n-element set S, in how many ways can we choose an unordered pair $\{X, Y\}$ of subsets of S such that $X \cup Y = S$?
5. Given an n-element set S, in how many ways can we choose an ordered triple (X, Y, Z) of subsets of S such that $X \cup Y \cup Z = S$?
6. Given an n-element subset S, in how many ways we can choose an ordered triple (X, Y, Z) of pairwise disjoint sets such that $X \cup Y \cup Z = S$?
7. In how many ways can we distribute 50 apples to 10 students?
8. In how many ways can we distribute 50 apples to 10 students if the first student has to get at least 5 apples?
9. In how many ways can we distribute 50 apples to 10 students if the first student has to get at most 10 apples?
10. In how many ways can we distribute 50 apples to 10 students if the first student has to get at least 5 apples and the second student has to get some number of apples divisible by 7?
11. In how many ways can we distribute 50 apples to 10 students if the first three students have to get at most 5 apples each?
12. In how many ways can we distribute 50 apples to 10 students if not all the apples have to be distributed?
13. Every card in a deck has a picture of one shape — circle, square, or triangle, which is painted in one of three colors — red, blue, or green. Furthermore, each color is applied in one of three shades — light, medium, or dark. The deck has 27 cards, with every shape-color-shade combination represented. A set of three cards from the deck is called *complementary* if all of the following statements are true:

    (i) Either each of the three cards has a different shape or all three of the cards have the same shape.
    (ii) Either each of the three cards has a different color or all three of the cards have the same color.
    (iii) Either each of the three cards has a different shade or all three of the cards have the same shade.

    How many different complementary three-card sets are there?
14. Determine the number of ways to paint each slice of an n-sliced pizza ($n \geq 3$) in one of k colors ($k \geq 4$) so that no two adjacent squares are of the same color.
15. How many 4-tuples (a, b, c, d) of non-negative integers satisfy the relation:
$$a + b + c + d \leq 10?$$
16. How many subsets of the set $\{1, 2, 3, 4, ..., 30\}$ have the property that the sum of the elements of the subset is greater than 232?
17. Let n be an odd positive integer. The unit squares of an $n \times n$ chessboard are colored alternately black and white, with the four corners colored black. What is the smallest possible value for n for which it is possible to cover all the black squares with non-overlapping figures congruent to ?
18. Determine the number of non-decreasing sequences of length 5 whose all terms belong to the set $\{1, 2, 3, \ldots, 10\}$.

19. A number of n tennis players take part in a tournament in which each of them plays exactly one game with each of the others. If x_i and y_i denote the number of wins, respectively, losses of the i-th player, prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2.$$

20. In each cell of $n \times n$ $(n \geq 2)$ there is one integer. All rows of this table are different. Prove that there exists a column after whose removal the remaining table satisfies that all of its rows are different.

21. Define a domino to be an ordered pair of distinct positive integers. A proper sequence of dominos is a list of distinct dominos in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not both appear for any i and j. Let D_{40} be the set of all dominos whose coordinates are no larger than 40. Find the length of the longest proper sequence of dominos that can be formed using the dominos of D_{40}.

22. Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled 1, 2, ..., $2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequence of *steps*: at each step one of the lamps is switched (from on to off or from off to on).

    Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off.

    Let M be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

    Determine the ratio N/M.