National Bulgarian Mathematical Olympiad 1997
Regional Round

(1) Find all positive integers $a$, $b$, and $c$ such that the roots of the equations

\[ x^2 - 2ax + b = 0 \]
\[ x^2 - 2bx + c = 0 \]
\[ x^2 - 2cx + a = 0 \]

are positive integers.

(2) Let the convex quadrilateral $ABCD$ be inscribed in a circle, $F = AC \cap BD$ and $E = AD \cap BC$. If $M$ and $N$ are the midpoints of $AB$ and $CD$, prove that

\[ \frac{MN}{EF} = \frac{1}{2} \left| \frac{AB}{CD} - \frac{CD}{AB} \right|. \]

(3) Prove that the equation

\[ x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0 \]

has no solutions in rational numbers.

(4) Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = f(x^2 + \frac{1}{4})$ for all $x \in \mathbb{R}$.

(5) Let $K_1$ and $K_2$ be unit squares with centers $M$ and $N$ such that $MN = 4$, two of the sides of $K_1$ are parallel to $MN$ and one of the diagonals of $K_2$ lies on the line $MN$. Find the locus of midpoints of the segments $XY$ where $X$ and $Y$ are interior points of $k_1$ and $k_2$, respectively.

(6) Find the number of all non-empty subsets of the set $S_n = \{1, 2, \ldots, n\}$ which do not contain two consecutive integers.
National Bulgarian Mathematical Olympiad 1997
Final Round

(1) Consider the polynomials \( P_n(x) = \sum_{i=0}^{k} \binom{n}{3i+2} x^i \) where \( n \geq 2 \) and \( k = \lfloor \frac{n-2}{3} \rfloor \).

(a) Prove that \( P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x) \).
(b) Find all integers \( a \) such that \( P_n(a^3) \) is divisible by \( 3^{\lfloor \frac{n-2}{3} \rfloor} \) for all \( n \geq 2 \).

(2) Let \( M \) be the centroid of \( \triangle ABC \). Prove the inequality

\[ \sin \angle CAM + \sin \angle CBM \leq \frac{2}{\sqrt{3}}. \]

(a) if the circumcircle of \( \triangle AMC \) is tangent to the line \( AB \);
(b) for any \( \triangle ABC \).

(3) Let \( n \) and \( m \) be positive integers and \( m + i = a_i b_i^2 \) for \( i = 1, 2, \ldots, n \), where \( a_i \) and \( b_i \) are positive integers and \( a_i \) is square-free. Find all \( n \) for which there exists \( m \) such that \( a_1 + a_2 + \cdots + a_n = 12 \).

(4) Let \( a, b, \) and \( c \) be positive numbers such that \( abc = 1 \). Prove the inequality

\[ \frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}. \]

(5) Let \( BM \) and \( CN \) be the angle bisectors in \( \triangle ABC \) and let ray \( MN \) intersect the circumcircle of \( \triangle ABC \) at point \( D \). Prove that

\[ \frac{1}{BD} = \frac{1}{AD} + \frac{1}{CD}. \]

(6) Let \( X \) be a set of \( n + 1 \) elements, \( n \geq 2 \). The ordered \( n \)-tuples \( (a_1, a_2, \ldots, a_n) \) and \( (b_1, b_2, \ldots, b_n) \) consisting of distinct elements of \( X \) are called “disjoint” if there exist distinct indices \( i \) and \( j \) such that \( a_i = b_j \). Find the maximal number of \( n \)-tuples an two of which are “disjoint”.

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