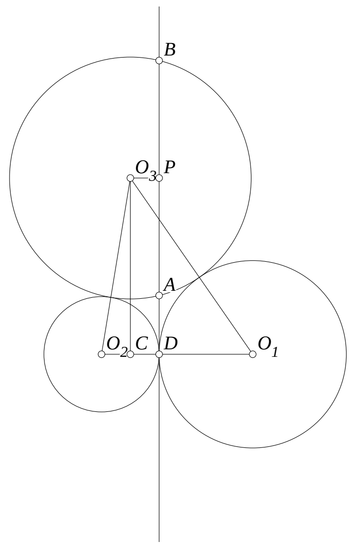
Russian Math Circle Solutions

April 15, 2009

1. Without loss of generality, assume that $R_1 > R_2$. (If they are equal, the problem is easy.) In the following figure, let O_1, O_2 and O_3 be the centers of the circles with radii R_1, R_2 and R_3, respectively. Let A and B be the endpoints of the secant in question, and drop perpendiculars from O_3 to AB and to O_1O_3, at P and C, respectively. D is the tangent point of the circles centered at O_1 and O_2.



If we denote by x the length of O_3P, it's easy to see that:

$$AB = 2\sqrt{R_3^2 - x^2}.$$

Since $O_2O_3 = R_2 + R_3$ and $O_1O_3 = R_1 + R_3$, we can use the Pythagorean theorem twice to obtain two equations for the length of O_3C:

$$\begin{aligned} O_3C^2 &= (R_2 + R_3)^2 - (R_2 - x)^2 \\ O_3C^2 &= (R_1 + R_3)^2 - (R_1 + x)^2 \end{aligned}$$

Setting these equal, we can solve for x and substitute that into the equation for AB, above.

After some algebra, we obtain:

$$x = \frac{(R_1 - R_2)R_3}{R_1 + R_2}.$$

The final substitution (again, after some algebra) yields:

$$AB = 4R_3\sqrt{\frac{R_1R_2}{(R_1+R_2)}}.$$

2. If we add all the numbers on all the cards, we obtain $1 + 2 + \ldots + 20 = 210$. Since there are 5 cards and each has the same total, the numbers on each card must sum to $210/5 = 42$.

   The remaining numbers not mentioned in the problem consist of:

   $$1, 5, 7, 8, 9, 12, 13, 14, 16 \text{ and } 20.$$

   The given information yields the following partial sums on the five cards:

   Card 1: $17 + 2 = 19$; 23 more required
   Card 2: $6 + 11 = 17$; 25 more required
   Card 3: $15 + 3 = 18$; 24 more required
   Card 4: $10 + 18 = 28$; 14 more required
   Card 5: $4 + 19 = 23$; 19 more required

   The number 20 has to go on Card 2 since all other sums with 20 will not work:

   Card 2: $6 + 11 + 20 + 5 = 42$

   Remaining numbers:
   $$1, 7, 8, 9, 12, 13, 14, 16.$$

   The only way to get 14 for Card 4 from these numbers is $13 + 1$:

   Card 4: $10 + 18 + 13 + 1 = 42$

   Remaining numbers:
   $$7, 8, 9, 12, 14, 16.$$

   Continue in the same way: $7 + 12 = 19$ for Card 5, then $14 + 9 = 25$ for Card1, and finally $8 + 16$ for Card 3. The final answer:

   Card 1: $17 + 2 + 9 + 14 = 42$
   Card 2: $6 + 11 + 20 + 5 = 42$
   Card 3: $15 + 3 + 8 + 16 = 42$
   Card 4: $10 + 18 + 13 + 1 = 42$
   Card 5: $4 + 19 + 7 + 12 = 42$

3. There are probably a lot of ways to do this. What follows is one possibility.

   Obviously there is no solution up to 12345678910, so the final palindrome must contain some zeroes.

   If there is a solution, N will consist of exactly k digits, ending in 987654321. Let "$A \ldots Z321$" be the digits of this N (where although the alphabet consists of 26 letters, we just label the first few "ABC" and the last ones "XYZ", and the ”...” represents perhaps millions of digits). The digit A, of course, cannot be zero.

Earlier in the palindrome appears $A0000\ldots000$ with $k-1$ zeros. Clearly strings of $k-1$ zeros are the longest ones in the final palindrome, and there are only a few of them. If A, for example, were 3, there will be three of them: $3000\ldots000$, $2000\ldots000$ and $1000\ldots000$.

Unfortunately, all of the strings of $k-1$ zeroes will look like: $A000\ldots000A$, with possible values of A ranging from 1 to 9, and they will all be different, so there can't be a palindrome if $A > 1$. Thus there must be a single string like: $1000\ldots0001$ in the exact center of the palindrome, but it is obvious that it is surrounded like this:

$$\ldots9991000\ldots00011000\ldots$$

but this is not palindromic.

4. Call the unknown 5-digit number $EDCBA$, where A, B, C, D and E are numbers between 0 and 9.

$$ABCDE \times 4 = EDCBA.$$

A must be even since it is $4 \times E$ (with a possible carry away). A cannot be zero, since we don't start 5-digit numbers with zero. If A is 4 or larger, the product would have six digits, so $A = 2$:

$$2BCDE \times 4 = EDCB2.$$

Since we're multiplying the lead 2 by 4, E is at least 8, so E is 8 or 9. But $E \times 4$ yields a 2 in the unit's digit, so $E = 8$:

$$2BCD8 \times 4 = 8DCB2.$$

When we multiply the 8 and 4 in the unit's place, $8 \times 4 = 32$, so there's a carry of 3. $4 \times D$ will be even, but adding the carry gives odd, or B is odd. But if B is 3 or larger, there will be a carry into the 10,000's place, so $B = 1$:

$$21CD8 \times 4 = 8DC12.$$

Now $4 \times D + 3$ yields a 2-digit number with a 1 in the unit's place, and the only possibilities for D are 2 and 7. D can't be 2, since it is at least 4 (4×1 in the 1000's place). So $D = 7$:

$$21C78 \times 4 = 87C12.$$

$C \times 4$ with a carry of 3 yields a number ending in C. $C = 9$ is the only digit that works, and if you put in 9, the whole thing works out:

$$21978 \times 4 = 87912.$$

Thus the answer to the original problem is 87912. Divide that by 4 andyou reverse the digits.