Russian Math Circle Problems

April 14, 2009

Instructions: Work as many problems as you can. Even if you can't solve a problem, try to learn as much as you can about it.

1. Two large jars contain 1 liter of water each. Half of the water from the first jar is poured into the second one; then one third of the water from the second jar is poured into the first one; then one fourth of the water from the first jar is poured into the second one; and so on. How much water is contained in each jar after repeating this operation 100 times?

2. If $n > 1$ and there are n people at a party, show that at least two of the people know the same number of people (who are at the party).

3. Two of the diagonals of a convex equilateral pentagon are perpendicular. If one of the interior angles of the pentagon is 100 degrees, compute the measures of all the other interior angles.

4. The Fibonacci sequence is defined as follows:

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-1} + F_{n-2}, \text{ for } n \geq 3. \end{aligned}$$

   The Fibonacci numbers F_a, F_b and F_c form an increasing arithmetic sequence. If $a + b + c = 2000$, compute a.

5. Let $n = ABC$ be a three-digit number, where A, B and C are the three digits. Compute the largest possible value of:

$$n/(A + B + C).$$

6. The number $85^9 - 21^9 + 6^9$ is divisible by an integer between 2000 and 3000. Compute the value of that integer.

7. The sequence $1, 2, 4, 5, 10, 11, 22, 23, 46, 47, \ldots$ is formed as follows:

   Begin with 1 and alternately add 1 to obtain the next number, and double the result to obtain the next. The thousandth term will be of the form:

$$3 \cdot 2^k - 1.$$

   Compute k.

8. Let n be a 5-digit number and let q and r be the quotient and remainder when n is divided by 100. For how many values of n is $(q + r)$ divisible by 11?