Russian Math Circle Solutions

April 14, 2009

Instructions: Work as many problems as you can. Even if you can't solve a problem, try to learn as much as you can about it.

1. Two large jars contain 1 liter of water each. Half of the water from the first jar is poured into the second one; then one third of the water from the second jar is poured into the first one; then one fourth of the water from the first jar is poured into the second one; and so on. How much water is contained in each jar after repeating this operation 100 times?

   Solution: Make a list of the first few steps and the result is obvious:

Step	Jar 1	Jar 2
0	1	1
1	$1/2$	$3/2$
2	1	1
3	$3/4$	$5/4$
4	1	1
5	$5/6$	$7/6$
6	1	1

   This can be formally solved by induction. At even-numbered steps, there is a liter in each bottle; on an odd-numbered step, $1/n$ of the water is moved from jar 1 to jar 2, so after that move, there is $(n-1)/n$ in jar 1 and $(n+1)/n$ in jar 2. The next step moves $1/(n+1)$, which is $1/n$ liter back, so we're back to one liter in each. The question asks about the state after 100 steps, and since 100 is even, the two jars will contain an equal amount of water.

2. If $n > 1$ and there are n people at a party, show that at least two of the people know the same number of people (who are at the party).

   Solution: This problem can be solved by the pigeon-hole principle. Since there are n people at the party, any one person can know either 0, 1, 2, $\ldots$, $n-1$ other people. There are n possibilities here so at first glance, it appears that each person can know a different number, but if somebody (call him/her person A) knows 0 people it is impossible for someone else to know $n-1$ people (since that person would have to know person A).

3. Two of the diagonals of a convex equilateral pentagon are perpendicular. If one of the interior angles of the pentagon is 100 degrees, compute the measures of all the other interior angles.

   Solution: Let the pentagon be $ABCDE$.

   If the perpendicular diagonals meet at a vertex, say vertex A, then $\angle CAD = 90°$ and $AC < CD$; then in triangle ABC, since $AB = CB = CD$, $\angle B$ will be less than $60°$ and base $\angle BAC$ will be greater than $60°$. By similar reasoning, $\angle EAD$ will be greater than $60°$, so $\angle EAB$ will be reflex, which is a contradiction.

Assume AC is perpendicular to BE, intersecting at P. Draw CE. Clearly $AP = PC$ and $BP = PE$. Then $ABCE$ is a rhombus, so $BC = EC = ED = DC$ (and triangle CDE is equilateral). If $\angle B = 100°$, the other angles of the pentagon are $140°$, $60°$, $160°$, $80°$. If $\angle BCD = 100°$, the $\angle BCE = 40°$, sor $\angle AED = 140° + 60° = 200°$, which is a contradiction.

4. The Fibonacci sequence is defined as follows:

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-1} + F_{n-2}, \text{ for } n \geq 3. \end{aligned}$$

   The Fibonacci numbers F_a, F_b and F_c form an increasing arithmetic sequence. If $a + b + c = 2000$, compute a.

   Solution: It's not hard to see that $(a, b, c) = (n, n + 2, n + 3)$, except for the sequence (F_2, F_3, F_4), since F_1 happens to be equal to F_2.

   Since $a + (a + 2) + (a + 3) = 2000$, $a = 665$.

5. Let $n = ABC$ be a three-digit number, where A, B and C are the three digits. Compute the largest possible value of:

$$n/(A + B + C).$$

   Solution: Let k be the solution. Since $n/(A + B + C) = k$, we have:

$$n = kA + kB + kC = 100A + 10B + C.$$

   This implies that $k \leq 100$, but $k = 100$ is possible if $B = C = 0$ and $A \neq 0$.

6. The number $85^9 - 21^9 + 6^9$ is divisible by an integer between 2000 and 3000. Compute the value of that integer.

   Solution: In general, $a^k - b^k$ is divisible by $a - b$, so $85^9 - 21^9$ is divisible by $85 - 21 = 64 = 2^6$, and $6^9 = 2^9 \cdot 3^9$. Thus the entire term is divisible by 2^6. For the same reason, $21^9 - 6^9$ is divisible by $21 - 6 = 15$, which has a common factor of 5 with 85. If k is odd, $a^k + c^k$ is divisible by $a + c$. The exponent 9 is odd, so $85^9 + 6^9$ is divisible by $85 + 6 = 91 = 7 \cdot 13$, and 21 is divisible by 7.

   Therefore, $85^9 - 21^9 + 6^9$ is divisible by $2^6 \cdot 5 \cdot 7 = 2240$.

7. The sequence $1, 2, 4, 5, 10, 11, 22, 23, 46, 47, \ldots$ is formed as follows:

   Begin with 1 and alternately add 1 to obtain the next number, and double the result to obtain the next. The thousandth term will be of the form:

$$3 \cdot 2^k - 1.$$

Compute k.

Solution: First look at the first few even terms of the series:

$$\begin{aligned} 2 &= 3 \cdot 2^0 - 1 \\ 5 &= 3 \cdot 2^1 - 1 \\ 11 &= 3 \cdot 2^2 - 1 \\ 23 &= 3 \cdot 2^3 - 1 \\ 47 &= 3 \cdot 2^4 - 1 \end{aligned}$$

If this pattern continues, it is clear that the answer will be: $k = 1000/2 - 1 = 499$.

A simple induction shows us that the pattern continues: If at some stage we are at $3 \cdot 2^k - 1$, doubling that yeilds $3 \cdot 2^{k+1} - 2$, and add 1 to get the next even term. Thus every two terms, the exponent k increases by 1.

8. Let n be a 5-digit number and let q and r be the quotient and remainder when n is divided by 100. For how many values of n is $(q + r)$ divisible by 11?

   Solution: Note that:

   $$n = 100q + r = 99q + (q + r).$$

   Thus n is divisible by 11 exactly when $(q + r)$ is divisible by 11. Thus all we need to do is to count the number of 5-digit numbers that are divisible by 11.

   Unless $n \geq 10000$ we will not have a 5-digit number, so the answer is given by:

   $$\left\lfloor \frac{99999}{11} \right\rfloor - \left\lfloor \frac{10000}{11} \right\rfloor = 9090 - 909 = 8181.$$