Anyone Got Change for a Dollar?

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Change for a Dollar

1. How many ways are there to make change for a dollar?

2. What if you are allowed to use at most 2 nickels?

3. What if you are allowed to use at most 3 of each coin?

4. In all of the above, what if the order of passing the coins matters, so that for instance QQQDDN is different from QQQDND and so on.

Sum of Squares

5. How many ways are there to write a given positive integer as the sum of four squares? (Should we include 0 here?)

6. What if you include squares of negative numbers? What if order matters?

Sum of Triangles

7. Can every positive integer be written as the sum of three triangular numbers?

8. How many ways are there to write a given positive integer as the sum of four triangular numbers? What if order matters? (Should we include 0 here?)

Binary and Beyond

It's possible to write some numbers, like 5, as the Sum Of at least two Consecutive Positive Integers: 2+3. We'll call this a SOCPI representation.

9. Some numbers have more than one SOCPI representation, like 15 = 1+2+3+4+5 = 4+5+6 = 7+8 = … is that all of them?

10. Find a number that has exactly two SOCPI, or explain why one doesn't exist. Same question with three, or four, or k SOCPI representations.

11. What numbers have no SOCPI representations?

12. What is a formula for determining how many SOCPI representations n has?

Some numbers, like 5, can be written as the Sum of Distinct Powers of Two: $2^2 + 2^0$. Distinct here means that you can use each power of 2 at most once. Numbers like 5 are OK, too: 2^2. You don't have to use more than one power of 2. We'll call these ways of writing a number SDP2 representations.

13. Find a formula for determining how many SDP2 representations a number has.

Every number, like 5, can be represented as the Sum of (not necessarily distinct) Powers of 2: $5 = 2^2 + 2^0 = 2^1 + 2^1 + 2^0 = 2^1 + 2^0 + 2^0 + 2^0 = 2^0 + 2^0 + 2^0 + 2^0 + 2^0$. So 5 has four SP2 representations.

14. Find a formula for determining how many SP2 representations a number has.

15. For what k does there exist a number n with exactly k SP2 representations?

Every number, like 5, can be represented as the Sum of at most 2 copies of (not necessarily distinct) Powers of 2: $5 = 2^2 + 2^0 = 2^1 + 2^1 + 2^0$. (The other examples from the previous section are no longer allowed, because there are more than 2 copies of one of the powers of 2). We'll call these S2P2 representations.

16. Is there a number with exactly 1 S2P2 representation? Why or why not?

17. For what values of k does there exist a number with exactly k S2P2 representations?

18. Construct a formula, or a method of determining, the number of S2P2 representations of any given positive integer n.

Some generalization ideas:

19. What if we allow all integers, not just positive integers? (This especially applies to SOCPI more than to powers of 2.)

20. What if we use powers of 3, or of 10, instead of powers of 2?