San Jose Math Circle December 13, 2008

SYSTEMS OF NUMERATION

For base ten as well as for any base b, where b is a positive integer greater than 1, the following theorem holds:

The Fundamental Theorem of Systems of Numeration: Every natural number n can be uniquely written as

$$n = a_m b^m + a_{m-1} b^{m-1} + a_{m-2} b^{m-2} + \cdots + a_2 b^2 + a_1 b + a_0$$

where a_0, a_1, $\ldots$, a_m and m are integers satisfying the following conditions: $0 \leqslant a_i < b$ for all $i = 0, 1, 2, \ldots, m$, $a_m \neq 0$ and $m \geqslant 0$.

Notation and terminology: We abbreviate

$$n = a_m b^m + a_{m-1} b^{m-1} + a_{m-2} b^{m-2} + \cdots + a_2 b^2 + a_1 b + a_0$$

as $n = a_m a_{m-1} \ldots a_1 a_0$ and we say that n is written in base b. The numbers 0, 1, 2, $\ldots$, $b-2$, $b-1$ are called digits in base b. The numbers 1, 2, 3, $\ldots$, $b-1$ are called significant digits.

If in a certain problem one deals with multiple bases, then the base is added in subscript to the right of the number. For example, 17 in base 3 is written as 17_3 or $17_{(3)}$. Unless specified by the context, numbers without subscript are considered to be decimal (in base 10).

History: The base-10 system is the most commonly used today. The binary system (base 2) is used to perform integer arithmetic in almost all digital computers. Modern computers today also use the octal (base 8) and hexadecimal (base 16) systems. A base-5 system has been used in many cultures for counting; the system was based on the number of fingers of the human hand. The Yuki tribe in Northern California still uses a quaternary (base-4) system, by counting the spaces between the fingers rather than the fingers themselves. Base-12 systems (duodecimal) have been popular because multiplication and division are easier than in base 10. They have been used extensively by the British, where 12 was a common unit of measurement (12 inches in a foot, 12 pennies in a shilling, etc). Twelve is also used as unit for analog and digital printing; printer resolutions like 360, 600, 720, 1200, 1440 dpi are everywhere. The Mayan civilization used a base-20 system, possibly originating from the number of a person's fingers and toes. The sexagesimal system (base 60) was used by Sumerians and their successors in Mesopotamia and survives today in our systems of time (hence the division of an hour into 60 minutes and a minute into 60 seconds) and angular measure (a degree is divided into 60 minutes and a minute into 60 seconds). The Chinese calendar still uses today a base-60 system to denote years.

Converting from base b to base 10:

(i) The conversion of a whole number from base b to base 10 uses the fundamental theorem:

$$(a_m a_{m-1} \dots a_1 a_0)_b = a_m b^m + a_{m-1} b^{m-1} + \dots + a_1 b + a_0$$

For example:

$$621_8 = 6 \cdot 8^2 + 2 \cdot 8 + 1 = 401$$
$$23010_5 = 2 \cdot 5^4 + 3 \cdot 5^3 + 0 \cdot 5^2 + 1 \cdot 5 + 0 = 1630$$

(ii) If the number in base b is negative, use the exact same formula:

$$-(a_m a_{m-1} \dots a_1 a_0)_b = -(a_m b^m + a_{m-1} b^{m-1} + \dots + a_1 b + a_0)$$

For example,

$$-5312_6 = -(5 \cdot 6^3 + 3 \cdot 6^2 + 1 \cdot 6 + 2) = -1196$$

(iii) Finally, decimal numbers in base b are converting to base 10 using:

$$(a_m a_{m-1} \dots a_1 a_0 . c_1 c_2 c_3 \dots)_b = a_m b^m + a_{m-1} b^{m-1} + \dots + a_1 b + a_0 + c_1 b^{-1} + c_2 b^{-2} + c_3 b^{-3} + \dots$$

For example,

$$11.011_2 = 1 \cdot 2 + 1 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = \frac{27}{8} = 3.375$$

Converting from base 10 to an arbitrary base b:

For converting a positive integer n to an arbitrary base b, use the following algorithm:

$$\begin{aligned} n &= b \cdot q_0 + a_0 & 0 \leqslant a_0 < b \\ q_0 &= b \cdot q_1 + a_1 & 0 \leqslant a_1 < b \\ q_1 &= b \cdot q_2 + a_2 & 0 \leqslant a_2 < b \\ \vdots \quad & \quad \vdots & \vdots \\ q_{m-2} &= b \cdot q_{m-1} + a_{m-1} & 0 \leqslant a_{m-1} < b \\ q_{m-1} &= b \cdot 0 + a_m & 0 \leqslant a_m < b \end{aligned}$$

Then $n = (a_m a_{m-1} \ldots a_1 a_0)_b$.

For example, suppose we want to convert 641 from base 10 to base 7. Using the above algoritm we get:

$$\begin{aligned} 641 &= 7 \cdot 91 + 4 \\ 91 &= 7 \cdot 13 + 0 \\ 13 &= 7 \cdot 1 + 6 \\ 1 &= 7 \cdot 0 + 1 \end{aligned}$$

Hence $641 = 1604_7$.

Operations with numbers in different bases: The addition, subtraction, multiplication, and division rules for whole numbers in base b are the same as the ones for base 10, provided one determines first the addition and multiplication tables in the new base.

Base 10 problems

1. (*2006 AMC 8, ♯24*) In the multiplication problem below, A, B, C, and D are different digits. What is $A + B$?

$$\begin{array}{r} ABA \\ \times \quad CD \\ \hline CDCD \end{array}$$

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 9

2. (*1993 National Sprint, ♯26*) Alice leaves Denver, driving at a constant speed. After a while, she passes a mile marker displaying a two-digit number. One hour later, she passes a second marker with the same two digits in reverse order. In another hour, she passes a third marker with the same two digits separated by a zero. What is the rate of Alice's car in miles per hour?

3. (*1994 Olympiad Romania, 2nd round, 6th grade*) Find the four-digit number $\overline{abbc}$ (in base 10) if $\overline{abbc} - \overline{abb} - \overline{ab} - a = 1775$.

4. Solve the equation: $\overline{xy} + \overline{yz} + \overline{xz} = 246$.

5. Let $A = \overline{abcabc}$. Show that A cannot be a perfect square.

6. (*1940 Gazeta Matematica, vol. 46*) Does there exist a perfect square of the form $\overline{aabbcc}$?

7. (*1930 Gazeta Matematica, vol. 36*) Show that the numbers 25, 1225, 112225, 11122225, $\ldots$, $\underbrace{11\ldots1}_{n-1}\underbrace{22\ldots2}_{n}5$ are perfect squares.

8. If $A = \underbrace{33\ldots3}_{666}$ and $B = \underbrace{66\ldots6}_{333}$, then what is the product AB?

9. (*1999 National Target, ♯1*) When a two-digit number is multiplied by 5, the result is a three-digit number. When the digit 7 is written after the resulting three-digit number, the new four-digit number is 1281 greater than the original two-digit number. What is the original number?

10. (*1962 IMO, proposed by Poland*) Find the smallest positive integer N satisfying the following two conditions:

    (i) The units digit of N is 6.

    (ii) If the units digit of N is moved first, then the number obtained is 4 times the initial number.

Problems with bases other than 10

11. Let 0, 1, 2, 3, 4, 5, 6 be the digits in base 7. Determine the addition and multiplication tables for base 7. Use the tables to find 632+45 and 24×65 in base 7.

+	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

×	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

12. Let 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, α, and β be the digits in base 12. Determine the addition and multiplication tables for base 12.

$+$	0	1	2	3	4	5	6	7	8	9	α	β
0												
1												
2												
3												
4												
5												
6												
7												
8												
9												
α												
β												

$\times$	0	1	2	3	4	5	6	7	8	9	α	β
0												
1												
2												
3												
4												
5												
6												
7												
8												
9												
α												
β												

13. *(1957 AMC 12, ♯19)* The base of the decimal number system is ten, meaning, for example, that $123 = 1 \cdot 10^2 + 2 \cdot 10 + 3$. In the binary system, which has base two, the first five positive integers are 1, 10, 11, 100, 101. The numeral 10011 in the binary system would then be written in the decimal system as:

(A) 19 (B) 40 (C) 10011 (D) 11 (E) 7

14. *(1960 AMC 12, ♯16)* In the numeration system with base 5, counting is as follows: 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, ... The number whose description in the decimal system is 69, when described in the base 5 system, is a number with:

(A) two consecutive digits (B) two non-consecutive digits
(C) three consecutive digits (D) three non-consecutive digits (E) four digits

15. *(1956 AMC 12, ♯31)* In our number system the base is ten. If the base were changed to four, you would count as follows: 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, ... The twentieth number would be:

(A) 20 (B) 38 (C) 44 (D) 104 (E) 110

16. *(2007 Olympiad Romania, 1st Round, 5th grade)*

   (a) Calculate: $71_9 - 61_8 + 51_7 - 41_6 + 31_5 - 21_4$.

   (b) Solve for x and y: $51_x + 71_y = 10^2$.

17. Solve the following equation for x:

$$55_6 = 44_5 - 33_4 + 22_3 - 11_2 + x_{10}$$

18. Solve the equation:

$$12_x + 36_y = 34_{10}$$

19. The number $\overline{2x}$ is written in base $x + 2$. If the decimal representation of the number is 19, find the unknown basis.

20. Find the number $\overline{xyz}$ in base 10 if:

$$23_x + 45_y + 16_z = 1001001_2$$

and x and y are distinct odd integers.

21. Find the digits x and y for which:

$$1111_2 + 1111_3 + 1111_4 + 1111_5 + 1111_6 + 1111_7 + 1111_8 + 1111_9 = \overline{xyxyxy}_7 - 91_{10}$$

22. A three digit number $\overline{xyz}$ in base 7 when written in base 9 becomes $\overline{zyx}$. What is the decimal representation of the number?

23. Find a three-digit positive integer n whose base-7 representation is $\overline{xyy}$ and base-6 representation is $\overline{yxx}$.

24. *(2005 Harker Math Invitational, Individual Contest, ♯10)* A base-two numeral consists of 15 digits all of which are ones. This number when tripled and written in base two, contains how many digits?

25. *(2004 Harker Math Invitational, Individual Contest, ♯17)* In a strange land they have a slightly different place value system than our base ten system. For example, $4 \times 6 = 30$ and $4 \times 7 = 34$. Based on this system give the value of $5 \times 4 \times 7$.

26. *(1987 AMC 12, ♯16)* A cryptographer devises the following method for encoding positive integers. First, the integer is expressed in base 5. Second, a 1-to-1 correspondence is established between the digits that appear in the expressions in base 5 and the elements of the set $\{V, W, X, Y, Z\}$. Using this correspondence, the cryptographer finds that three consecutive integers in increasing order are coded as VYZ, VYX, VVW, respectively. What is the base ten expression for the integer coded as XYZ?

(A) 48 (B) 71 (C) 82 (D) 108 (E) 113

27. *(2003 AMC 10 A, ♯20)* A base-10 three digit number n is selected at random. Which of the following is closest to the probability that the base-9 representation and the base-11 representation of n are both three-digit numerals?

(A) 0.3 (B) 0.4 (C) 0.5 (D) 0.6 (E) 0.7

28. *(2007 AMC 12 B, ♯21)* The first 2007 positive integers are each written in base 3. How many of these base-3 representations are palindromes? (A palindrome is a number that reads the same forward and backward.)

(A) 100 (B) 101 (C) 102 (D) 103 (E) 104

29. *(2004 AMC 12 A, ♯25)* For each integer $n \geqslant 4$, let a_n denote the base-n number $0.\overline{133}_n$. The product $a_4 a_5 \cdots a_{99}$ can be expressed as $\frac{m}{n!}$, where m and n are positive integers and n is as small as possible. What is the value of m?

(A) 98 (B) 101 (C) 132 (D) 798 (E) 962

30. In what base does the following hold:

$$25 \times 314 = 10274\,?$$

31. Show that in any base larger than 7 the number 1367631 is a perfect cube.