Russian Math Circle Homework Problems

October 30, 2008

Instructions: Work as many problems as you can. Even if you can't solve a problem, try to learn as much as you can about it.

1. Let x, y, z, v, and w be nonnegative integers. If

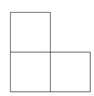
$$2^{x+y+z} + 2^{y+z+v} + 2^{z+v+w} + 2^{v+w+x} = 1089$$

   show that $x + y + z + v + w$ is a perfect square.

2. Write the number 100^{100} as a sum of four perfect cubes.

3. A group of airplanes is based on a small island. The tank of each plane holds just enough fuel to take it halfway around the world. Any desired amount of fuel can be transferred from the tank of one plane to the tank of another while the planes are in flight. The only source of fuel is on the island, and we assume that there is no time lost in refueling either in the air or on the ground. What is the smallest number of planes that will ensure the flight of one plane around the world on a great circle, assuming that the planes have the same constant ground speed and rate of fuel consumption and that all planes return safely to the island base?

4. An ell is an L-shaped tile made from three 1×1 squares (see picture below). For what positive integers a, b is it possible to completely tile an $a \times b$ rectangle only using ells? For example, it is clear that you can tile a 2×3 rectangle with ells, but (draw a picture) you cannot tile a 3×3 rectangle with ells.



5. Does there exist a pentagon in space, all of whose sides are equal and all of whose angles are 90 degrees?

6. Two circles in the plane intersect at point A and B. Starting at A, two moving points P and Q start traveling each along one of the circles, in the same sense and at a constant speed. The two points return to A simultaneously after completing one revolution. Show:

   - P, Q and B are always collinear;
   - There exists a point S in the plane such that the distances SP and SQ are equal at all times.

7. Let O be the point of intersection of the diagonals of a convex quadrilateral $ABCD$. What is the smallest possible area of $ABCD$ if the area of the triangle AOB is 4 and the area of the triangle COD is 9?

8. In a triangle with area S, drop perpendiculars from the middle of each edge to the other two edges. A hexagon is formed inside the triangle; calculate its area.