

San Jose Math Circle – Graph Coloring

October 2008

- (1) A *coloring* of a map is a coloring of the different regions (or countries, however you view your map) in such a way that regions that share parts of their boundary get different colors. (If two countries only share a few points on their borders, such as Utah and New Mexico on a map of the US, they can get the same color.) The *chromatic number* of a map is the minimum number of colors we need to color it. Construct (many) maps with chromatic number 1, 2, 3, 4, and 5.
- (2) A *graph* is a mathematical concept consisting of *nodes* (which you can just think of as points) and *edges* between these nodes (which you can think of as curves connecting the node points). If there is an edge between two nodes, we say that these nodes *share* the edge, or that the two nodes are *adjacent*. Every map gives rise to a graph (called the *dual graph* of the map) in the following sense: each region in the map gives rise to a node in the graph, and two nodes share an edge if the corresponding regions share part of their boundaries. A *coloring* of a graph is a coloring of its nodes such that any two adjacent nodes get distinct colors. The *chromatic number* of a graph is the smallest number of colors needed to color the graph.
 - (a) Convince yourself that a map coloring corresponds exactly to a coloring of the corresponding graph.
 - (b) Go through your examples in (1) again in terms of the graphs that come with your maps.
 - (c) What do you notice about the graphs that arise from maps?
- (3) Here are four classes of graphs:
 - The *null graph* N_n consists of n nodes and no edges.
 - The *complete graph* K_n consists of n nodes with all possible edges between them.
 - The *line graph* L_n consists of n nodes with edges that form a line segment.
 - The *cycle* C_n consists of n nodes with edges that form a circle.

Compute the chromatic numbers of these graphs. What do you notice about

- (a) the leading coefficients?
 - (b) the second leading coefficients?
 - (c) the constant term?
 - (d) the highest degree?
- (4) Given a graph G and a positive integer k , let $c_G(k)$ be the number of colorings of G that use at most k colors. Compute $c_G(k)$ for the four classes mentioned above.
 - (5) Fix an edge e in a graph G . Denote by $G \setminus e$ be the graph you get from G by *deleting* e , and by $G \cdot e$ the graph you get from G by *contracting* e (i.e., identifying the two nodes that share e). Find a relationship of $c_G(k)$, $c_{G \setminus e}(k)$, and $c_{G \cdot e}(k)$.
 - (6) What graphs do you obtain by continuously deleting and contracting edges of a given graph G ? Use this observation and the identity you found in (5) to prove that $c_G(k)$ is a polynomial in k .
 - (7) Use a similar reasoning to prove your observations from (3).
 - (8) Experiment with the numbers $c_G(-1)$ for different examples of graphs G . Can you guess what they count? (Hint: look at *acyclic orientations* of G , i.e., give each edge a direction, in such a way that you can't see any coherently oriented cycle.) Try to prove your assertion.