San Jose Math Circle
Combinatorics of the Chessboard – Ivan Matić

1. Initially, number 1 is 2006 times written on the blackboard. A student is playing the following game: at each step he replaces two of the numbers from the blackboard by the quarter of their sum. After 2007 steps only one number will remain at the blackboard. Prove that this last number must be greater than or equal to $1/2008$.

2. Is it possible to tile 6×6 board by 1×4 rectangles? What about 10×10 board?

3. A corner square is removed from the 8×8 chessboard. Can the rest be tiled by 1×3 rectangles?

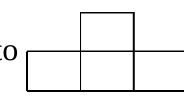
4. A frog is jumping on a 8×8 chessboard. At each step the frog jumps from one unit square to one of the squares that is adjacent to the previous position of the frog (squares are *adjacent* if they share an edge). Is it possible for the frog to start from the lower-left corner of the chessboard, visit each unit square exactly once, and finish its trip at the upper-right corner of the chessboard? Justify your answer!

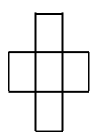
5. A cube $3 \times 3 \times 3$ is made of cheese and consists of 27 small cubical cheese pieces arranged in the $3 \times 3 \times 3$ pattern. A mouse is eating the cheese in such a way that it starts at one of the corners and eats smaller pieces one by one. After he finishes one piece, he moves to the adjacent piece (pieces are adjacent if they share a face). Is it possible that the last piece mouse has eaten is the central one?

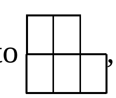
   Remark: Pieces don't fall down if a piece underneath is eaten first.

6. A chocolate has a rectangular shape of the form $m \times n$. The left-top corner is poisoned. Two players are playing the following game: Players alternate the moves and in each turn a player choses one piece of the chocolate which is not previously eaten and eat that piece together with all pieces that are down and right to the chosen piece. A player who eats the poisonous piece loses the game. Which player has a winning strategy?

7. Determine the greates number of figures congruent to  that can be placed in a grid 7×7 (without overlaping) such that each figure covers exactly 4 unit squares.

8. Is it possible to tile 8×8 board with figures congruent to ? What about 10×10 board?

9. Determine the maximal number of figures  that can be placed in a grid 6×6 without overlaping.

10. If a $5 \times n$ rectangle can be tiled with n pieces congruent to , prove that n is even.

11. Two players A and B play the following game on an infinite chessboard. Initially, all cells are empty. Player A starts the game and each his move consists of writing the letter X in some empty cell of the board. After his move, player B writes the letter O in some other cell, etc. The winner is the player who manages to put his sign in

   (a) all cells of some of the rectangles 1×5, 5×1 or all cells of some 2×2 square.

   (b) 11 cells that are consecutive and lie on some of the horizontal, vertical or diagonal line.

   Prove that no player has a winning strategy.