Dynamic Geometry

Tom Rike

San Jose Math Circle April 23, 2008

1 Background

The topic for this talk comes from a paper by Ross Finney in *Mathematics Magazine*, volume 43 (1970), pp 177-185. I have not added anything original and am only demonstrating what is shown in the article. Another source for similiar material is the New Math Library book by I.M. Yaglom, *Geometric Transformations, Volume 1.* In this book, the material is presented very carefully from the beginning with justification.

2 Some Lemmas

Lemma 1. If isosceles triangles ZMX and YMW have right angles at M, then YX and ZW are perpendicular and congruent. See Figure 1.
Lemma 2. If Z and X are the centers of squares that lie on sides of ABC, built towards the exterior of ABC, and if M is the midpoint of the third side, then ZMX is isosceles and has a right angle at M. See Figure 2.

3 Theorems

Theorem 1. If X, Y and Z, W are opposite pairs of centers of squares on the sides of a quadrilateral that lie towards the quadrilateral's exterior, then YX and WZ are perpendicular and congruent. See Figure 3.
The theorem is quite general in that the quadrilateral need not be convex, or even simple. To place squares on a quadrilateral with no obvious interior, traverse the quadrilateral in one direction, laying squares to the right. See Figure 4.
Problem 1. Prove that Theorem 1 also holds if the squares all lie towards the interior of the quadrilateral.
Problem 2. Prove that if the quadrilateral in Theorem 1 is a parallelogram then $ZXWY$ is a square.
Theorem 2. If squares are constructed on the sides of a triangle towards the triangle's exterior, then the segment joining two of the centers is perpendicular and congruent to the segment joining the third center to the vertex opposite it. See Figure 5.
Problem 3. Prove If squares are constructed on the sides of a triangle towards the triangle's exterior, that the three lines joining the centers of the squares to the vertices opposite them are concurrent. See Figure 6.
Theorem 3. The vertices Z, X, W, and Y of equilateral triangles built on the sides of a quadrilateral, and lying alternately towards the interior and the exterior of the quadrilateral, are themselves the vertices of a parallelogram. See Figure 7.

Theorem 4. If Z, X, W, and Y are vertices of similar triangles appropriately arranged on the sides of a quadrilateral, then $ZXWY$ is a parallelogram. See Figure 8.
For Theorem 5 we assume the angles may have a variety of measures which agree modulo 360, measuring them in a counterclockwise fashion.
Theorem 5. Let BZC and AXC be nondegenerate similar triangles (with vertices corresponding in the order given) constructed both towards the exterior or both towards the interior of arbitrary triangle ABC. Let the angles BZC and CXA have measure β. Let M be the point in the plane that is equidistant from A and B and that is located so that angle BMA has a measure 2β. Then $MZ = MX$. See Figure 9.

Corollary 1 Suppose that 30-60-90 triangles are built on two sides of an arbitrary triangle towards the exterior. Let Z and X denote the outer vertices of these triangles, and let M be the midpoint of the remaining side of the given triangle. Then ZMX is equilateral. If instead, the 30-60-90 triangles lie toward the interior of the given triangle, ZMX is still equilateral. See Figure 10.

Corollary 2 (**Napoleon's Theorem**) The centers X, Z, and M of equilateral triangles constructed on the sides of an arbitrary triangle, and lying towards the triangles exterior, are themselves the vertices of an equilateral triangle. See Figure 11.

Corollary 3 Suppose that equilateral triangles are built on the sides of an arbitrary triangle, two towards its exterior and one towards its interior. Let M be the center of the inner one, and Z and X be the apexes of the outer ones. Then ZMX is an isosceles triangle with a 120° angle at M. See Figure 12.

To conclude let me state the three opening problems of Yaglom's book, *Geometric Transformations, Volume 1.* (My numbering.)

Problem 4. Construct a triangle, given the three points in the plane that are the outer vertices of equilateral triangles constructed outward on the sides of the desired triangle.

Problem 5. Construct a triangle, given the three points in the plane that are the centers of the squares constructed outward on the sides of the desired triangle.

Problem 6. Construct a heptagon (polygon with 7 sides), given the seven points that are the midpoints of its sides.

Yaglom then goes on to outline solutions using the "usual *school book* methods." Then he says, "These solutions of the three problems are rather artificial; they involve drawing certain auxiliary lines (and how does one know which lines to draw?) and they demand considerable ingenuity. The study of isometries enables one to pose and solve the following more general problem . . . can literally be solved in one's head," and which solves all three problems.

Problem 7. Construct an $n-$gon given the n points that are the outer vertices of isosceles triangles constructed outward on the sides of the desired $n-$gon (with the sides for bases), and such that these isosceles triangles have vertex angles $\alpha_1, \alpha_2 \ldots, \alpha_n$.

If you have comments, questions or find glaring errors, please contact me by e-mail at the following address: tricycle222@mac.com

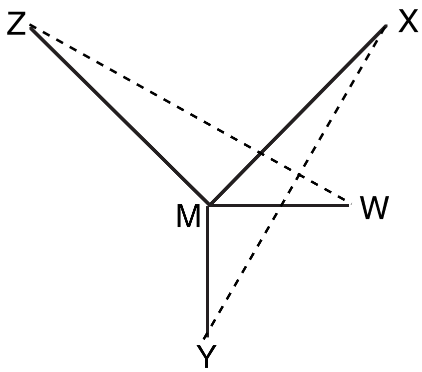

Fig. 1

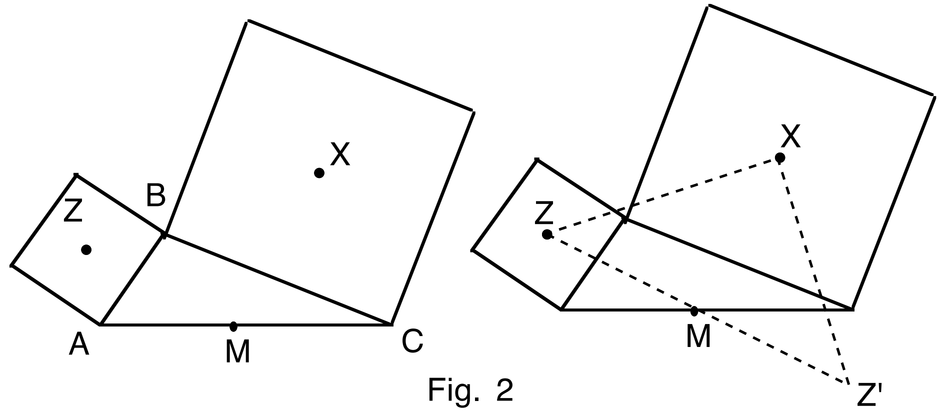

Fig. 2

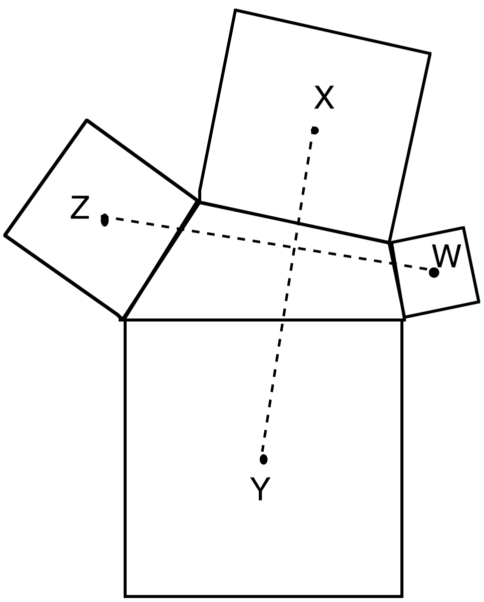

Fig. 3

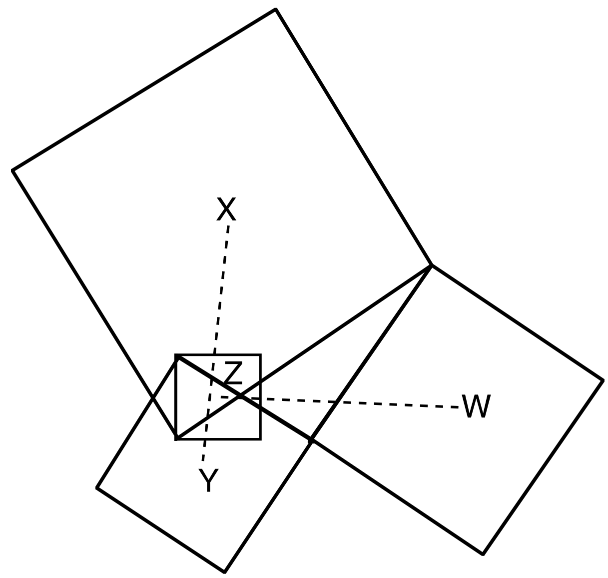

Fig. 4

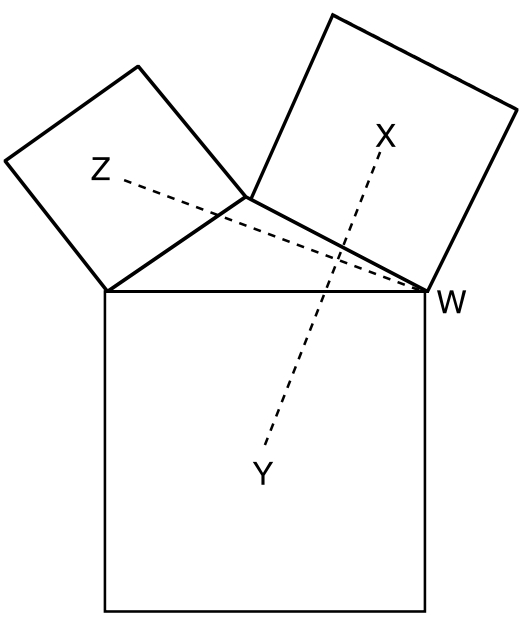

Fig. 5

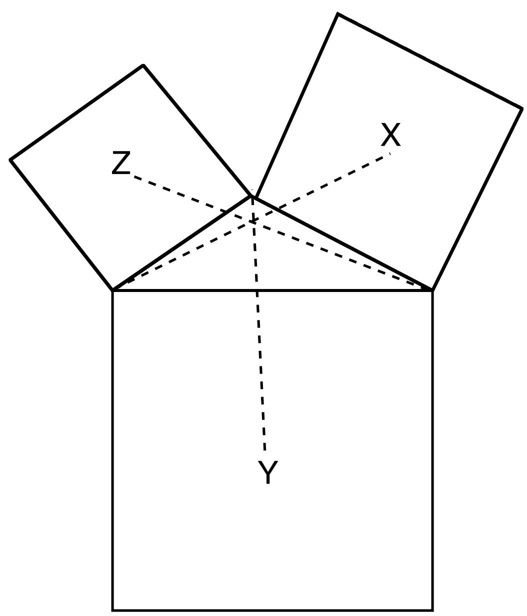

Fig. 6

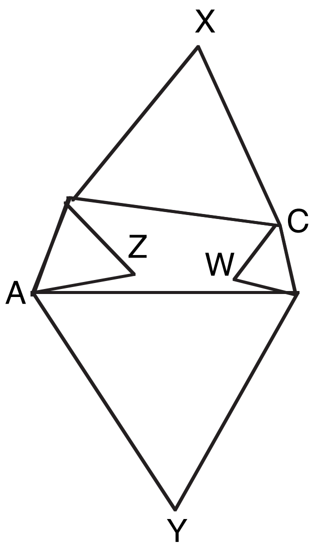

Fig. 7

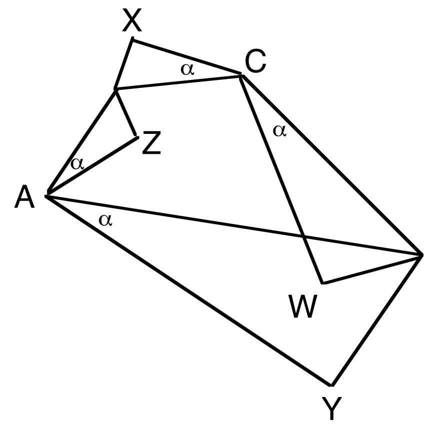

Fig. 8

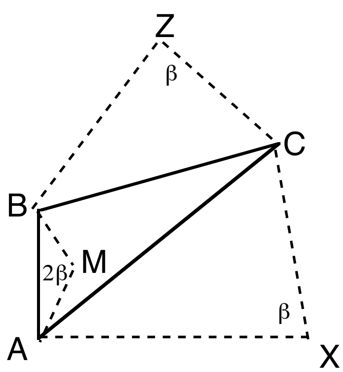

Fig. 9

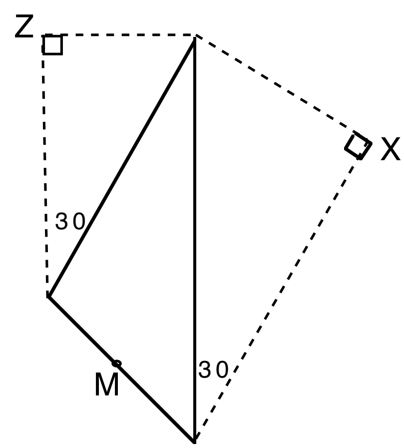

Fig. 10

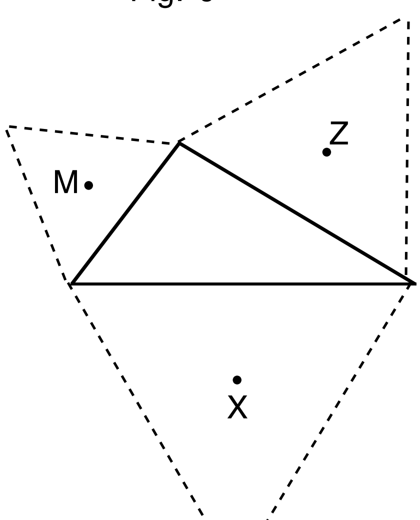

Fig. 11

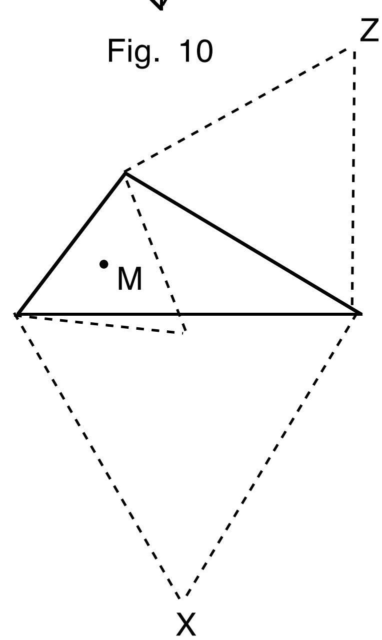

Fig. 12