San Jose Math Circle
Combinatorics and Mathematical Games
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1. A frog is jumping on a 8×8 chessboard. At each step the frog jumps from one unit square to one of the squares that is adjacent to the previous position of the frog (squares are *adjacent* if they share an edge). Is it possible for frog to start from the lower-left corner of the chessboard, visit each unit square exactly once, and finish its trip at the upper-right corner of the chessboard? Justify your answer!

2. A cube $3 \times 3 \times 3$ is made of cheese and consists of 27 small cubical cheese pieces arranged in the $3 \times 3 \times 3$ pattern. A mouse is eating the cheese in such a way that it starts at one of the corners and eats smaller pieces one by one. After he finishes one piece, he moves to the adjacent piece (pieces are adjacent if they share a face). Is it possible that the last piece mouse has eaten is the central one?

   Remark: Pieces don't fall down if a piece underneath is eaten first.

3. There are three prisoners in a prison. A warden has 2 red and 3 green hats and he has decided to play the following game: He puts the prisoners in a row one behind the other and on the had of each prisoner he puts a hat. The first prisoner in the row can't see any of the hats, the second prisoner can see only the hat at the had of the first one, and the third prisoner can see the hats of the first two prisoners. If some of the prisoners tells the color of his own hat, he is free; but if he is wrong, the warden will kill him. If a prisoner remain silent for sufficiently long, he is returned to his cell. Of course, each of them would like to be rescued from the prison, but if he isn't sure about the color of his hat, he won't guess.

   After noticing that second and third prisoner are silent for a long time, first prisoner (the one who doesn't see any hat) has concluded the color of his hat and told that to the warden. What is the color of the hat of the first prisoner? Explain your answer! (All prisoners know that there are 2 red and 3 green hats in total and all of them are good at mathematics.)

4. There are 5 piles of coins arranged in a row. The piles are numbered as 0, 1, 2, 3, 4, respectively. The pile 1 contains 10, pile 2 - 20, pile 3 - 30, and pile 4 - 40 coins. Two players A and B play the following game: A starts and the players alternate their moves, in each move a player has to choose one of the piles 1, 2, 3, 4 that contains at least one coin and move several (but at least one) coins to the pile that is immediately to the left. The player who moves the last coin is a winner. Who has a winning strategy and what is the strategy?

5. Two players A and B play the following game with a coin on a 8×8 chessboard: At the beginning the coin is in the lower-left corner of the board and in each step a player can move the coin to the adjacent cell that is to the down-right, right, upper-right, up, upper-left to the original position of the coin. The winner is the player who makes the last move. Player A starts the game. Which player has a winning strategy? Determine the strategy.

6. A group of mathematicians is lost in a forest. The forest has a shape of an infinite strip that is 1 mile wide. Prove that they can choose a path that will guarantee them a way out and that is at most $2\sqrt{2}$ miles long.

7. Two players A and B play the following game on an infinite chessboard. Initially, all cells are empty. Player A starts the game and each his move consists of writing the letter X in some empty cell of the board. After his move, player B writes the letter O in some other cell, etc. The winner is the player who manages to put his sign in

   (a) all cells of some of the rectangles 1×5, 5×1 or all cells of some 2×2 square.

   (b) 11 cells that are consecutive and lie on some of the horizontal, vertical or diagonal line.

   Prove that no player has a winning strategy.