Polyhedra

by Matt Beck

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1 Introduction

In the following text, all polytopes and polygons are convex.

Definition: A set is convex if any line segment connecting two points of the set is also included in this set.

A polytope is the generalization of a polygon to higher dimensions. To be precise, we can **define** a polytope to be the smallest convex set containing a given set of finitely many points in d-dimensional space.

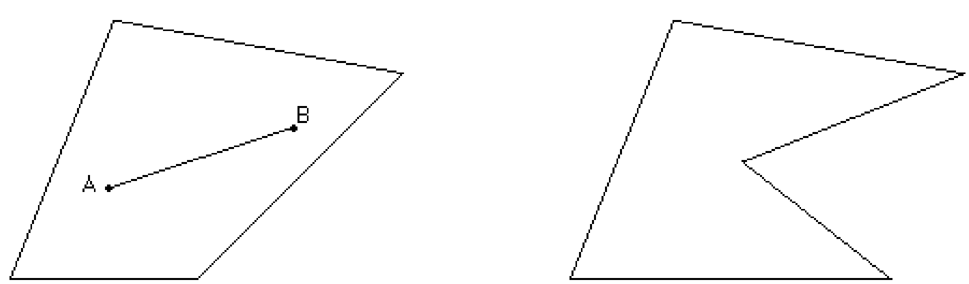


Figure 1: Convex vs concave polygon

Now, what is an edge? What is a vertex? How do we define them? In 2D: We have an intuition of it, but we need a rigorous and simple definition to identify edges. Read the following definition carefully (maybe more than one time) and notice the picture:

Defn 1. If we can find a line that intersects the polygon in a line segment and that has all remaining points of the polygon lying on one side of the line, then we call the line segment (i.e., the intersection of the polygon with the line) an **edge**.

Defn 2. If we can find a line that intersects the polygon in a point and that has all remaining points of the polygon lying on one side of the line, then we call the point (i.e., the intersection of the polygon with the line) a **vertex**.

We can make a similar definition for what we call the **faces** of a d-dimensional polytope; now lines have to be replaced by planes (in 3-d) and hyperplanes (in higher dimensions). Thus a face of a polytope P is the intersection of P with a hyperplane, which has the remaining points of P all on one side.

2 Euler and face numbers

Let's look at face numbers.

- f_0 = number of vertices of the polytope (a vertex is 0 dimensional)
- f_1 = number of edges of the polytope (an edge is 1 dimensional)
- f_2, f_3, ..., f_{n-1} = number of 2, 3, ..., n-1-dimensional faces of the polytope.

Euler's Theorem for n-dimensional polytopes states that:

$$\text{f\_0 - f\_1 + f\_2 - f\_3 + f\_4 ...} \pm \text{f\_n-1} = \begin{cases} 2 & n = 2k+1 \\ 0 & n = 2k \end{cases}, k \in \mathbb{N}$$

In 2 dimensions, Euler's Theorem just states that we need to have as many edges as vertices. Euler's Theorem is not too hard to prove for n=3 but the proof gets very involved for n >= 4.

Here's a famous research question: Given a sequence of n integers (an n-tuple), determine if you can construct a polytope that has these n numbers as face numbers. Examples:

1. (4,4) Yes, square. Euler's Theorem says: For the 2-tuple of face numbers, f_0=f_1 for a polygon associated with those numbers to exist. (n,n) for 2-D polytopes. So in 2-d we need to require that f_0 = f_1 >= 3.

2. (4,6,4) Tetrahedron. In 3-d, there is a classification theorem that says precisely which 3-dim vectors form face numbers and which do not.

3. For 4-D, no such classification theorem has yet been found! Here's your chance to be famous!

3 Regular polytopes

Regular polytopes have some nice properties. Some of them are:

1. All edges have equal length.

2. All angles between adjacent edges are equal.

3. Each vertex has the same number of edges eminating from it

4. ... and many others!

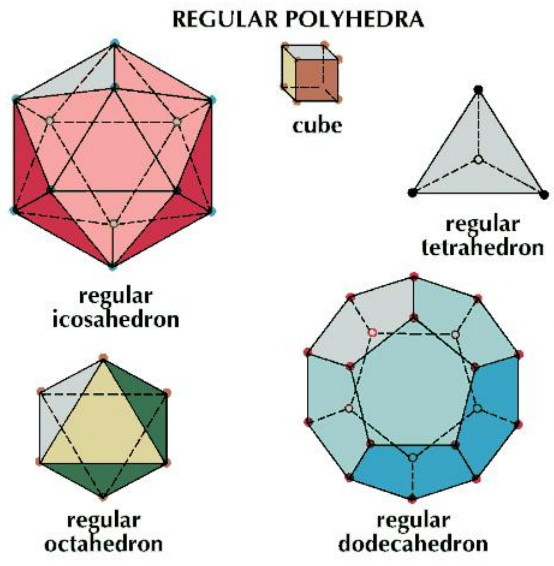
For 2-D, there are infinitely many regular polytopes. Ex:

- (3,3) - equilateral triangle, (4,4) - Square, (5,5) Regular pentagon, and so on...

- For 3-D, curiously enough, there are only 5 regular polytopes. Why? Because of angle conflicts. Think of a regular polytope in 3-D in terms of the above 2nd and 3rd property of regular polytopes. To understand why there are only finitely many, answer this question: Can you construct a polytope having 6 edges coming together at a vertex, and 60 degrees angles? Why can't you have angles smaller than 60 degrees?

The *five regular polytopes in 3-D* are:

1. Tetrahedron
2. Cube
3. Octahedron
4. Icosahedron
5. Dodecahedron



References

[1] Below are the sources of the images of this text:

- The third, last image: http://www.britannica.com/ebc/art-53380/Regular-polyhedra
- The first and second image: http://mathforum.org/sum95/math_and/poly/polygon.html