Egyptian Fractions

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1 What are Egyptian fractions?

An Egyptian fraction is a sum of one or more simple fractions with 1 as the numerator, and any whole positive integer as the denominator. Because Egyptians did not have any notation for fractions with a number other than 1 as the numerator, they had to express numbers like $\frac{2}{5}$ as a sum of unit fracions $(= \frac{1}{k}, k \in \mathbb{N})$.

For example,

$\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$

1.1 What is one way of writing a regular simple fraction $\frac{a}{b}$ as a sum of unit fractions?

Tip: Keep subtracting the biggest unit fraction at every step, until you reach 0.

1.2 Question: Is there always an Egyptian fraction for every $\frac{a}{b}$?

To answer this question, it might be helpful if we look at continued fractions.

2 What are continued fractions?

2.1 A continued fraction is a fraction of the form:

$a_0 + \frac{1}{a1+\frac{1}{a2+\dots}}$, where $a_i \in \mathbb{N}, a_i \neq 0$(nonzero positive integers) with the exception of $a_0$which can be negative or zero.

Examples of continued fractions include:

$1 + \frac{1}{1+1}$, $-3 + \frac{1}{3+\frac{1}{6}}$, etc.

Let's take examples of simple fractions and write them as continued fractions. What do you notice?

2.2 What is your method for converting from a simple fraction to a continued fraction?

Tip: Take out the "whole number" of times that the denominator goes into the numerator (ex 3 goes three times into 10, with remainder 1). Let that be the whole number, and add to that the rest. Then, write what's left as $\frac{1}{something}$(flip it), and take out the whole again.

2.3 Question: Is there always a continued fraction for every $\frac{a}{b}$?

Answer: Yes. The "something" (from the tip above) always decreases (try to prove that!), so the number of steps is finite, which guarantees that we will have a continued fraction for every $\frac{a}{b}$.

3 Further in the talk, we looked at infinite series contained in continued fractions.

For example, the number $x = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}$. How to find the numerical value of x?

Tip: Look inside of the series and find x again!

$=> x = 1 + \frac{1}{x}$. We get a quadratic equation with two solutions (one negative and one positive), choose the positive solution because the other doesn't make sense, and there we have our numerical value for x!

Other examples of infinite continued fractions are:

$x = 1 + \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{1+...}}}}$. Here, if we look carefully, we see that $x = 1 + \frac{1}{2+\frac{1}{x}}$. Solve for x!

4 To go back to Egyptian fractions, how are they related to continued fractions? Do you see?

References

[1] Below are some useful links for those who want to know more about Egyptian fractions and continued fractions.

- http://www.mathcats.com/explore/oldegyptianfractions.html
- http://mathworld.wolfram.com/EgyptianFraction.html
- http://kevingong.com/Math/EgyptianFractions.pdf
- http://en.wikipedia.org/wiki/Egyptian_fraction
- http://en.wikipedia.org/wiki/Continued_fraction
- http://mathworld.wolfram.com/ContinuedFraction.html
- http://archives.math.utk.edu/articles/atuyl/confrac/