Russian-Style Math Circle Problems

All the following problems are taken from the individual round questions from ARML contests. They are of varying difficulty, and some are much easier than they look. You are not expected to work all of them in a single circle session; choose a couple of them that are interesting to you and work on them.

1. ARML 1999: A pro athlete played for 17 years and earned 72 million dollars. She was paid k million dollars each year where k is an integer and received an extra one million dollars each year her team made the playoffs. Compute the number of years her team made the playoffs.

   $72/17 = 4.23$, so $k = 4$. $4 \cdot 17 = 68$ and $72 - 68 = 4$, so she was in the playoffs 4 times.

2. ARML 1995: Find the largest prime factor of:

   $$3(3(3(3(3(3(3(3(3(3(3+1)+1)+1)+1)+1)+1)+1)+1)+1)+1)+1.$$

   This reduces to:

   $$3^{11} + 3^{10} + \cdots + 1 = \frac{3^{12}-1}{3-1}.$$

   $$\frac{3^{12}-1}{2} = \frac{(3^6+1)(3^6-1)}{2} = \frac{(3^2+1)(3^4-3^2+1)(3^3+1)(3^3-1)}{2} = \frac{10 \cdot 73 \cdot 28 \cdot 26}{2},$$

   so the largest prime factor is obviously 73.

3. ARML 1995: Compute the number of distinct planes passing through at least three vertices of a given cube.

   There are three kinds of planes: the six cube faces, the six that pass through 4 points (opposite edges), and the eight that pass through only 3 vertices, for a total of $6 + 6 + 8 = 20$.

4. ARML 1996: The roots of $ax^2 + bx + c = 0$ are irrational, but they are approximately 0.8430703308 and -0.5930703308. If a, b and c are integers, and $a > 0$, $|b| < 10$ and $|c| < 10$, find a, b and c.

   $-b/a \approx .843 - .593 = .25$, so $-b/a = 1/4$. Similarly $c/a \approx (-.593)(.843) \approx -0.5$. Thus $a = 4$, $b = -1$ and $c = -2$.

5. ARML 1997: Let a, b, c and n be positive integers. If $a + b + c = (19)(97)$ and $a + n = b - n = c/n$, compute the value of a.

   Since $b = a+2n$, $c = an+n^2$, then $2a+2n+an+n^2 = 2(1+n)+n(a+n) = (n+2)(a+n) = 19 \cdot 97$. Thus $n + 2 = 1, 19, 97$, or 1843. If $n + 2 = 1$ then $n < 0$ and if $n + 2 = 97$ or 1843, then $a < 0$. So $n + 2 = 19$ so $n = 17$ so $a + 17 = 97$ so $a = 80$.

6. ARML 1998: Compute:

   $$199919981997^2 - 2 \cdot 199919981994^2 + 199919981991^2.$$

   Let $x = 199919981991$. The expression is:

   $$(x+6)^2 - 2(x+3)^2 + x^2 = 36 - 18 = 18.$$

7. ARML 1999: Arrange the following products in increasing order from left to right:

$$1000! \quad (400!)(400!)(200!) \quad (500!)(500!) \quad (600!)(300!)(100!) \quad (700!)(300!).$$

Since they all have the same number of terms, it's clear that $1000!$ is the largest. Also clear is that:

$$(700!)(300!) > (500!)(500!) > (400!)(400!)(200!)$$

and

$$(700!)(300!) > (600!)(300!)(100!) > (400!)(400!)(200!).$$

We just need to compare $(600!)(300!)(100!)$ with $(500!)(500!)$.

$$\frac{(500!)(500!)}{(600!)(300!)(100!)} = \frac{(500 \cdot \ldots \cdot 401)(400 \cdot \ldots \cdot 301)}{(600 \cdot \ldots \cdot 501)(100 \cdot \ldots \cdot 1)} > (4/5)^{100}(3/1)^{100} = (12/5)^{100} > 1.$$

So:

$$1000! > 700!300! > 500!500! > 600!300!100! > 400!400!200!.$$

8. ARML 2000: If, from left to right, the last seven digits of $n!$ are 8000000, compute the value of n.

There have to be 6 factors of 5, so $5, 10, 15, 20, 25$ will do it. Thus $25 \leq n \leq 29$. We only need to look at the unit's digits of remaining factors and remember to omit 6 factors of 2, giving a final non-zero digit of $25!$ of 4. For $26!$ it will still be 4, but for $27!$ it will be 8, for $28!$, it will be 4, and finally, for $29!$ it will be 6, so $n = 27$ is the only solution.

9. ARML 2001: The vertices of a regular dodecagon (12-sided figure) are given by (x_i, y_i), for $i = 1, 2, 3, \ldots, 12$. if $(x_1, y_1) = (15, 9)$ and $(x_7, y_7) = (15, 5)$, compute:

$$\sum_{i=1}^{12}(x_i - y_i) = (x_1 - y_1) + (x_2 - y_2) + \cdots + (x_{12} - y_{12}).$$

The center is at $(15, 7)$. If $(15 + x, 7 + y)$ is on the dodecagon, so is $(15 - x, 7 - y)$. Add 6 copies of $8 + x - y$ and 6 copies of $8 - x + y$ and we obtain $8 \cdot 12 = 96$.

10. ARML 2003: Compute the largest factor of 1001001001 that is less than 10000.

$$1001001001 = 1001 \cdot 1000001 = 7 \cdot 11 \cdot 13 \cdot (10^6 + 1).$$

But $10^6 + 1 = (10^2 + 1)(10^4 - 10^2 + 1) = 101 \cdot 9901$. No combination of $7, 11, 13$ and 101 can generate a factor larger than 9901, so the answer is 9901. (It was pointed out at the circle that some argument must also be made that 9901 is not the product of a pair of factors that might be used together with the smaller factors to produce a larger factor. In fact, 9901 is prime.)