Russian-Style Math Circle, April 9, 2008 Set I, Solution

1. m non-empty boxes contain a total of mk boxes. Thus the total number of boxes including the original box is $mk+1$. Thus there are

$$mk+1-m=m(k-1)+1$$

empty boxes.

2. Suppose that there exist natural numbers m, n, and p such that $m^m+n^n=p^p$. Then $p^p>m^m$, $p^p>n^n$. If $p\le m$, then $p^p\le m^p\le m^m$, a contradiction. Assuming that $p\le n$ leads to a similar contradiction. Thus $p>m$ and $p>n$, and hence $p\ge m+1$ and $p\ge n+1$. Therefore $p^p\ge(m+1)^{m+1}=(m+1)(m+1)^m\ge 2(m+1)^m>2m^m$, i.e., $p^p>2m^m$. Similarly, $p^p>2n^n$. Hence $2p^p=p^p+p^p>2m^m+2n^n$, whence $p^p>m^m+n^n$, a contradiction.

3. Let $A(XYZ)$ denote the area of a triangle XYZ. Let also $AB=BC=AC=a$.

$$A(ABC)=A(APB)+A(BPC)+A(APC)=\frac{1}{2}\left(AB\cdot PF+BC\cdot PD+AC\cdot PE\right)=$$

$$\frac{a}{2}\left(PF+PD+PE\right)=\frac{1}{2}a^2\frac{\sqrt{3}}{2},\quad so\quad PF+PD+PE=\frac{a\sqrt{3}}{2}$$

Now let us draw trough point P lines parallel to the sides of the triangle ABC. Let the line parallel to BC intersect AB and AC at the points M and N, respectively; let the line parallel to AC intersect AB and BC at the points Q and R, respectively; and let the line parallel to AB intersect AC and BC at the points S and T, respectively. Triangles QMP, PTR, and SPN are all similar to ABC, thus they are equilateral. Hence PF, PD, and PE are their medians. Also, $AQPS$, $MBTP$, and $NPRC$ are parallelograms. Hence

$$BD=BT+TD=BT+\frac{1}{2}TR,\quad CE=CN+NE=PR+\frac{1}{2}SN=TR+\frac{1}{2}PN=TR+\frac{1}{2}RC,$$

and $AF=AQ+\frac{1}{2}QM=PS+\frac{1}{2}MP=PN+\frac{1}{2}BT=RC+\frac{1}{2}BT$. Therefore

$$BD+CE+AF=BT+\frac{1}{2}TR+TR+\frac{1}{2}RC+RC+\frac{1}{2}BT=\frac{3}{2}(BT+TR+RC)=\frac{3a}{2}.$$

Thus $\dfrac{PD+PE+PF}{BD+CE+AF}=\dfrac{a\sqrt{3}/2}{3a/2}=\dfrac{1}{\sqrt{3}}$.

4. If $b-a<0.1$, then $1+100(b-a)^2<2$. Thus a line segment whose length is less than 0.1 can contain at most one point. Therefore the distance between any two consecutive chosen points is greater than or equal to 0.1.

Let $a_1, a_2, \ldots, a_n$ be n chosen points and suppose that they are labeled in such a way that $0\le a_1<a_2<\ldots<a_n\le 1$. By the aforementioned property we must have: $a_2-a_1\ge 0.1$, $a_3-a_2\ge 0.1, \ldots, a_n-a_{n-1}\ge 0.1$. Adding all these inequalities we get

$a_n - a_1 \geq (n-1) \cdot 0.1$. On the other hand, $1 \geq a_n - a_1$. Thus $1 \geq (n-1) \cdot 0.1$. Therefore,
$n \leq 11$, so the largest possible number n does not exceed 11. Clearly, it's possible to find 11 points satisfying problem's conditions: take
$a_1 = 0,\ a_2 = 0.1,\ a_3 = 0.2, \ldots, a_{11} = 1$.
Hence the largest possible value of n is 11.

5. Clearly, the segment BO_3 is a diameter of k_1. Thus $\angle BAO_3 = 90^\circ$. Let D be the point of intersection of the line AB and k_2. Since $\angle BAO_1 = \angle DAO_2$ and $AO_1 = AO_2$, so isosceles triangles AO_1B and AO_2D are congruent, and so $AB = AD$. Therefore, O_3A is at the same time an altitude and median of the triangle BO_3D. Hence this triangle is isosceles and so $BO_3 = DO_3$. Since B belongs to k_3, so D must also belong to it. Thus the point B either coincides with the point P or with the point Q, i.e., the line AB either through P or through Q.

6. Let $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, $\{c_1, c_2, c_3\}$ be any partition of the set S. Let $A = a_1 \cdot a_2 \cdot a_3$, $B = b_1 \cdot b_2 \cdot b_3$, $C = c_1 \cdot c_2 \cdot c_3$. Then
$P^3 \geq A \cdot B \cdot C = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 8 \cdot 9 = 72 \cdot 72 \cdot 70 > 70^3$, i.e., $P > 70$. P is the largest of three composite numbers A, B, and C. Thus P is a composite number satisfying the inequality $P > 70$. This means that for any partition $P \geq 72$ (since 71 is a prime).
Let 's partition S into subsets {1, 8, 9}, {3, 4, 6}, {2, 5, 7}. Since for this partition $P = 72$, we're done.

7. Let $x_1, x_2, \ldots, x_n$ be a real solution of the system. It is clear that either all these numbers are positive or they are all negative. W.o.l.o.g., let's assume that they are all positive. For every k = 1, 2, … , n, we have

$\left(x_k - \dfrac{2}{x_k}\right)^2 \geq 2$, so $x_k^2 - 4 + \dfrac{4}{x_k^2} \geq 0$ and hence $x_k^2 + 4 + \dfrac{4}{x_k^2} \geq 8$, so that

$\left(x_k + \dfrac{2}{x_k}\right)^2 \geq 8$. But this implies that $2x_k = x_{k-1} + \dfrac{2}{x_{k-1}} \geq \sqrt{8} = 2\sqrt{2}$, i.e.,

$$\sqrt{2} \leq x_k \quad \text{and} \quad \frac{2}{x_k} \leq \frac{2}{\sqrt{2}} = \sqrt{2} \quad \text{for every } k = 1, 2, \ldots, n.$$

Adding all equations of the system, we now get:

$$n \cdot \sqrt{2} \leq x_1 + x_2 + x_3 + \ldots + x_n = \frac{2}{x_1} + \frac{2}{x_2} + \frac{2}{x_3} + \ldots + \frac{2}{x_n} \leq n \cdot \sqrt{2}$$

Obviously the equality can only be achieved if $x_1 = x_2 = \ldots = x_n = \sqrt{2}$. It is easy to see that these numbers indeed satisfy the given equation. Thus we have shown that the system has exactly two real solutions, i.e.,

$x_1 = x_2 = \ldots = x_n = \sqrt{2}$, and
$x_1 = x_2 = \ldots = x_n = -\sqrt{2}$.