Russian Math Circle, April 9, 2008. Problems, Set 1

1. We place k boxes in an empty box. In each of these k boxes we either place k new boxes or leave it empty. We keep this process going. Eventually, there are m boxes containing other boxes. Find the number of empty boxes.

2. Find all natural numbers x, y, and z satisfying the following equation.

$$x^x + y^y = z^z.$$

(Reminder: n is a natural number if it is a positive integer.)

3. Let ABC be an equilateral triangle and suppose that P is any point in its interior. Let D, E, and F be points on sides BC, CA, and AB, respectively, such that $PD \perp BC$, $PE \perp CA$, and $PF \perp AB$. Find the value of the quotient $\dfrac{PD+PE+PF}{BD+CE+AF}$.

4. Suppose that n points $P_1, P_2, \dots, P_n$ are chosen in the interval [0, 1] in such a way that every subinterval $[a, b]$ of [0, 1] contains no more than $1+100(b-a)^2$ of these points. What is the largest possible value of n?

5. Let k_1, k_2 be circles each of radius R with centers O_1 and O_2, respectively, and suppose that they are tangent to each other at a point A. Let k_3 be a circle of radius $2R$ with the center O_3 such that k_1 lies inside k_3 and is tangent to it a point B. Suppose further that k_2 and k_3 intersect at points P and Q. Prove that the line AB passes either through P or through Q.

6. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Suppose that we break S into 3 subsets each containing 3 elements. For each of these 3 subsets, we calculate the product of its elements, and choose the largest of these 3 numbers. We call this number P. (For example, we can take the subsets to be $\{1, 2, 9\}$, $\{3, 5, 8\}$, and $\{4, 6, 7\}$. For this arrangement, the products are 18, 120, and 168, and hence $P = 168$.) How should the 3 subsets be chosen in order to make P as small as possible?

7. Find all real solutions of the system of equations.

$$2x_2 = x_1 + \frac{2}{x_1}$$

$$2x_3 = x_2 + \frac{2}{x_2}$$

....

$$2x_n = x_{n-1} + \frac{2}{x_{n-1}}$$

$$2x_1 = x_n + \frac{2}{x_n}$$