

Russian Math Circle Problems

December 10, 2007

1. Suppose we are working in base-12, where the “digits” are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a , b , where a stands for 10 in base-10 and b stands for 11 in base-10. How many terminal zeros will there be if you expand $1a5!$ completely?
2. How many terminal zeroes are there in the expansion of $36! - 24!$ (we are back in base-10)?
3. Find the final digit of $7^{11}8^{135}$.
4. Show that $2222^{5555} + 5555^{2222}$ is divisible by 7.
5. Find the final digit of the number $1^2 + 2^2 + 3^2 + \dots + 99^2$.
6. The Fibonacci sequence is defined as follows: $F_0 = 0$, $F_1 = 1$, and if $n > 1$, $F_n = F_{n-1} + F_{n-2}$. The first few terms in the sequence are:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .

For which n is F_n divisible by 2? by 3? by 5? by 7? by 11?

7. Which entries in Pascal’s triangle are divisible by 2? by 3? by 7?
8. $ABCD$ is a quadrilateral (not necessarily convex) of area 1. K, L, M and N are the midpoints of the sides AB, BC, CD , and AD , respectively. Find the area of $KLMN$.
9. Alice has 12 cookies and Bob has 9 cookies. Charley, who has no cookies, pays Alice and Bob 42 cents to share their cookies. Each one of them eats one-third of the cookies. Bob says that he and Alice should split 42 cents evenly, and Alice thinks that she should get 24 cents and Bob should get 18 cents. What is the fair division of 42 cents between Alice and Bob?
10. Can the product of two consecutive natural numbers be equal to the product of two consecutive even numbers?
11. Suppose $f(x) = x^2 + 12x + 30$. Solve an equation

$$f(f(f(f(f(x)))))) = 0.$$