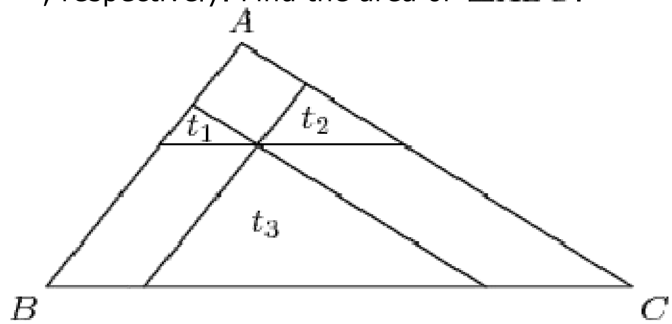
1. A point P is chosen in the interior of $\triangle ABC$ such that when lines are drawn through P parallel to the sides of $\triangle ABC$, the resulting smaller triangles t_1, t_2, and t_3 in the figure, have areas 4, 9, and 49, respectively. Find the area of $\triangle ABC$.



(‘84/3)

Solution: By the transversals that go through P, all four triangles are similar to each other by the AA postulate. Also, note that the length of any one side of the larger triangle is equal to the sum of the sides of each of the corresponding sides on the smaller triangles. We use the identity $K = \frac{ab\sin C}{2}$ to show that the areas are proportional (the sides are proportional and the angles are equal) Hence, we can write the lengths of corresponding sides of the triangle as $2x,\ 3x,\ 7x$. Thus, the corresponding side on the large triangle is $12x$, and the area of the triangle is $12^2 = \boxed{144}$.

2. The function f is defined on the set of integers and satisfies $f(n) = \begin{cases} n-3 & \text{if } n \geq 1000 \\ f(f(n+5)) & \text{if } n < 1000 \end{cases}$

Find $f(84)$.

(‘84/7)

Solution 1: Define $f^h = f(f(\cdots f(f(x))\cdots))$, where the function f is performed h times. We find that $f(84) = f(f(89)) = f^2(89) = f^3(94) = \ldots f^y(1004)$. $1004 = 84 + 5(y-1) \Longrightarrow y = 185$. So we now need to reduce $f^{185}(1004)$.

Let's write out a couple more iterations of this function:

$$f^{185}(1004) = f^{184}(1001) = f^{183}(998) = f^{184}(1003) = f^{183}(1000)$$
$$= f^{182}(997) = f^{183}(1002) = f^{182}(999) = f^{183}(1004)$$

So this function reiterates with a period of 2 for x. It might be tempting at first to assume that $f(1004) = 999$ is the answer; however, that is not true since the solution occurs slightly before that. Start at $f^3(1004)$:

$$f^3(1004) = f^2(1001) = f(998) = f^2(1003) = f(1000) = \boxed{997}$$

Solution 2: We start by finding values of the function right under 1000 since they require iteration of the function.

$$f(999) = f(f(1004)) = f(1001) = 998 \; f(998) = f(f(1003)) = f(1000) = 997$$
$$f(997) = f(f(1002)) = f(999) = 998 \; f(996) = f(f(1001)) = f(998) = 997$$

Soon we realize the $f(k)$ for integers $k < 1000$ either equal 998 or 997 based on it parity. (If short on time, a guess of 998 or 997 can be taken now.) If k is even $f(k) = 997$ if k is odd $f(k) = 998$. 84 has even parity, so $f(84) = 997$.

3. Assume that a, b, c, and d are positive integers such that $a^5 = b^4$, $c^3 = d^2$, and $c - a = 19$. Determine $d - b$. ('85/7)

Solution: It follows from the givens that a is a perfect fourth power, b is a perfect fifth power, c is a perfect square and d is a perfect cube. Thus, there exist integers s and t such that $a = t^4$, $b = t^5$, $c = s^2$ and $d = s^3$. So $s^2 - t^4 = 19$. We can factor the left-hand side of this equation as a difference of two squares, $(s - t^2)(s + t^2) = 19$. 19 is a prime number and $s + t^2 > s - t^2$ so we must have $s + t^2 = 19$ and $s - t^2 = 1$. Then $s = 10, t = 3$ and so $d = s^3 = 1000$, $b = t^5 = 243$ and $d - b = 757$.

4. Suppose that $x_i| < 1$ for $i = 1, 2, \ldots, n$. Suppose further that $x_1| + |x_2| + \cdots + |x_n| = 19 + |x_1 + x_2 + \cdots + x_n|$. What is the smallest possible value of n? ('88/4)

Solution: Since $x_i| < 1$ then

$x_1| + |x_2| + \cdots + |x_n| = 19 + |x_1 + x_2 + \cdots + x_n| < n$

So $n \geq 20$. We now just need to find an example where $n = 20$: suppose $x_{2k-1} = \frac{19}{20}$ and $x_{2k} = -\frac{19}{20}$; then on the left hand side we have $\left|\frac{19}{20}\right| + \left|-\frac{19}{20}\right| + \cdots + \left|-\frac{19}{20}\right| = 20\left(\frac{19}{20}\right) = 19$. On the right hand side, we have $19 + \left|\frac{19}{20} - \frac{19}{20} + \cdots - \frac{19}{20}\right| = 19 + 0 = 19$, and so the equation can hold for $n = 020$

5. In a shooting match, eight clay targets are arranged in two hanging columns of three targets each and one column of two targets. A marksman is to break all the targets according to the following rules:
1) The marksman first chooses a column from which a target is to be broken.
2) The marksman must then break the lowest remaining target in the chosen column.
If the rules are followed, in how many different orders can the eight targets be broken? (‘90/8)

Solution: Suppose that the columns are labeled A, B, and C. The question is asking for the number of ways to shoot at the bottom target of the columns, or the number of ways to arrange 3 As, 3 Bs, and 2 Cs in a string of 8 letters.

Of the 8 letters, 3 of them are As, making $\binom{8}{3}$ possibilities. Of the remaining 5, 3 are Bs, making $\binom{5}{3}$ possibilities. The positions of the 2 Cs are then fixed. Thus, there are $\binom{8}{3} \cdot \binom{5}{3} = 560$ ways of shooting all of the targets.

Alternatively, the number of ways to arrange 3 As, 3 Bs, and 2 Cs in a string of 8 letters is equal to $\frac{8!}{3! \cdot 3! \cdot 2!} = \boxed{560}$.

6. A positive integer is called ascending if, in its decimal representation, there are at least two digits and each digit is less than any digit to its right. How many ascending positive integers are there? (‘92/2)

Solution: Note that an ascending number is exactly determined by its digits: for any set of digits (not including 0, since the only position for 0 is at the leftmost end of the number, i.e. a leading 0), there is exactly one ascending number with those digits.

So, there are nine digits that may be used: $1, 2, 3, 4, 5, 6, 7, 8, 9.$ Note that each digit may be present or may not be present. Hence, there are $2^9 = 512$ potential ascending numbers, one for each subset of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

However, we've counted one-digit numbers and the empty set, so we must subtract them off to get our answer, $512 - 10 = \boxed{502}$.