Discrete Calculus

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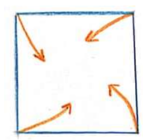
Motivation

Exercise 1

- It starts snowing at 11am and continues steadily throughout the day. A snowplow starts plowing at noon. It clears 2 miles of road in the first hour. How many miles of road can it clear in the second hour?
- Assume that the rate at which the snowplow clears the snow is constant (i.e., the greater the height of snow, the slower the snowplow moves)

Exercise 2

- Four dogs are in four corners of a square of side length l. Each dog starts running at speed v towards the dog immediately anti-clockwise to it. Even as the dogs change position, each maintains a bearing directly towards its neighboring dog.



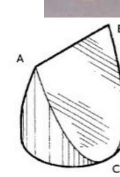
- With some imagination, you can see that the dogs will run in a spiral before they all meet in the center. How much time elapses before the group collision?

Exercise 3

- A percolator prepares coffee at 200ºF and pours it into a cup. After 10 minutes, the coffee has cooled down to 150ºF. After another 10 minutes, what is the temperature of the coffee?
- Assume that the temperature of the room 70ºF, and that the rate at which the coffee cools is proportional to the temperature difference between the coffee and the room (i.e., the greater the temperature difference, the faster the coffee cools)

Exercise 4

- You probably know that the volume of a cylinder is $\pi r^2 h$. This may be understood intuitively by imagining the cylinder to be a stack of coins, where each coin has area πr^2 and the entire stack has height h.



- How would you figure out the volume of a cylinder that has been sliced with two plane cuts, as shown in the adjoining figure?

Theory

Discrete-domain function

We are familiar with functions $f(x)$ where x is real. A **discrete-domain function** is similar except that x is an integer.

Only the domain is discrete, not the range. I.e., the value assumed by $f(x)$ is not constrained to be an integer.

Example: $f(x) = \frac{3^x - \sqrt{5}}{2}$, where $x \in \{\ldots - 3, -2, -1,\ 0,\ 1,\ 2,\ 3 \ldots\}$

Just to remind ourselves of the fact that x is an integer, we will henceforth the notation $f[x]$ instead of $f(x)$. (This is just a mnemonic: there no deep mathematical idea here.)

Discrete Integral

We are given a discrete-domain function $f[x]$, a starting value for x (call it integer a), and an stopping value for x (call it integer b).

We are interested in computing the sum $f[a] + f[a+1] + \cdots + f[b-2] + f[b-1]$

This sum is called "the **discrete integral** of the function f from a to b", and denoted by $\int_a^b f[x]$

Example:

- Suppose $f[x] = 2x - 1$, and the starting & stopping values are $a = 10$ & $b = 15$. Then

  $$\int_{10}^{15} f[x]$$

  $$= f[10] + f[11] + f[12] + f[13] + f[14]$$

  $$= (2.10 - 1) + (2.11 - 1) + (2.12 - 1) + (2.13 - 1) + (2.14 - 1)$$

  $$= 19 + 21 + 23 + 25 + 27$$

  $$= 115$$

Significance:

- Computing integrals is a problem that pops up again and again and again (and again!) in all sorts of disciplines:
  - physics, chemistry and modern approaches to biology
  - in every flavor of engineering
  - in all kinds of applied fields like biotechnology, prosthetics, medical instrumentation, geomatics, oceanography, forestry, mining, scuba diving, NASCAR racing …

- Given how commonly one has to compute integrals, it would be nice to have some general approach to it. This is where derivatives fit in (see next section).
- PS: The act of computing the integral is called **integration**. "Integral" is the noun, "integration" is the verb.

Discrete Derivative

We are given a discrete-domain function $f[x]$.

We are interested in computing a new discrete-domain function, which at each x, assumes the value $f[x+1]-f[x]$

This new function is called "the **discrete derivative** of function f", and denoted by $\Delta f[x]$

Example:

- Suppose $f[x]=x^2$. Then, at each x,

  $\Delta f[x]$

  $= f[x+1]-f[x]$

  $= (x+1)^2-x^2$

  $= x^2+2x+1-x^2$

  $= 2x+1$

Significance:

- The single biggest reason to gain expertise in derivatives is to help us compute integrals (see next section).
- Loose analogy: the single biggest to gain expertise in musical scales is to help us play songs.
- PS: The act of computing the derivative is called differentiation. "Derivative" is the noun, "differentiation" is the verb.

Some basic derivatives to memorize:

Basic $f[x]$	Corresponding $\Delta f[x]$
1	0
x	1
$x^{2\downarrow}$	$2x$
$x^{3\downarrow}$	$3x^{2\downarrow}$
$x^{4\downarrow}$	$4x^{3\downarrow}$
$x^{\mathrm{m}\downarrow}$	$mx^{(\mathrm{m-1})\downarrow}$
2^x	2^x
3^x	$2\,.\,3^x$
4^x	$3\,.\,4^x$
k^x	$(k-1).\,k^x$

Some important properties of derivatives

Scaling a function by a constant	Suppose $g[x] = c.f[x]$ Then $\Delta g[x] = c.\Delta f[x]$
Sum of functions	Suppose $h[x] = f[x] + g[x]$ Then $\Delta h[x] = \Delta f[x] + \Delta g[x]$
Product of functions	Suppose $h[x] = f[x]\,.\,g[x]$ Then $\Delta h[x] = f[x+1]\,.\,\Delta g[x] + g[x]\,.\,\Delta f[x]$

Fundamental Theorem of Discrete Calculus

The fundamental theorem of calculus tells us how we can use our expertise with discrete derivatives to help us compute discrete integrals

Problem set-up:

- You are given a discrete-domain function $f[x]$ and starting & stopping values a & b; and you are asked to compute the discrete integral $\int_a^b f[x]$

Solution approach:

- Guess another discrete-domain function $F[x]$ whose derivative $\Delta F[x]$ happens to be exactly the given function $f[x]$. There is no bullet-proof method to find $F[x]$: sometimes you just have to guess and check repeatedly.
- Assuming you can find such an $F[x]$, then you are home free: simply compute $F[b] - F[a]$, and offer it as the answer to the given problem $\int_a^b f[x]$

Surprising as it sounds, the above solution approach works. This is the whole point of the **Fundamental Theorem of Calculus**, which says:

> If $f[x] = \Delta F[x]$, then $\int_a^b f[x] = F[b] - F[a]$

The proof is quite straightforward:

$$\int_a^b f[x]$$

$= f[a] + f[a+1] + \cdots + f[b-2] + f[b-1]$

$= \Delta F[a] + \Delta F[a+1] + \cdots + \Delta F[b-2] + \Delta F[b-1]$

$= (F[a+1] - F[a]) + (F[a+2] - F[a+1]) + \cdots + (F[b-1] - F[b-2]) + (F[b] - F[b-1])$

$= F[b] - F[a]$ // because everything else cancels out – in a way that reminds us of a collapsing telescope!