Game Theory

Compiled by Raghu Subramanian – for San Jose Math Circle

Two Kinds of Games

When mathematicians say "games", they mean two different things:

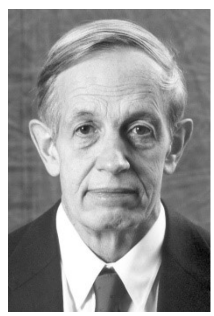
- Sequential games – players take turns
    - Examples: Tic-tac-toe, Chess, Nim, Chomp …
    - Lot of fun!

- Simultaneous games – players act simultaneously
    - Examples: Prisoner's Dilemma, Chicken, Rock Paper Scissors …
    - Tremendous applications to Economics, Politics, day-to-day life

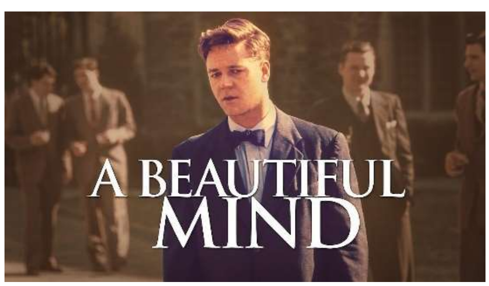
When mathematicians say "game ***theory***", they mean the second.

History

Game theory was invented by John Nash in his PhD thesis in 1950. In 1994, he got the Nobel Prize for it.



He was genius, but unfortunately battled with schizophrenia for most of his life. There is a great movie about him called "A Beautiful Mind" (2001), which won four Academy Awards.



Two-Player Games with Single Nash Equilibrium

The Prisoner's Dilemma

- Two bank robbers, Dave and Henry, have been arrested and are being interrogated in separate rooms.
- The police's case is rather weak: they can only prove the case against them if they can convince at least one of the robbers to betray his accomplice and testify to the crime.
- Each bank robber is faced with the choice either (a) to cooperate with his accomplice and remain silent, or (b) to betray the accomplice and testify for the prosecutor.

- If both remain silent, then the police will only be able to convict them on the lesser charge of "loitering", which is one year in jail each (1 year for Dave + 1 year for Henry).
- If one testifies and the other does not, then the one who testifies will go free (as part of the "deal" offered by the prosecutor) and the other will get 20 years in jail.
- If both testify, each will get five years in jail for being partly responsible for the robbery (5 years for Dave + 5 years for Henry).
- What should they do?

The Battle of Bismarck

- This was an actual battle fought in the south-west Pacific Ocean in the Second World War.
- General Kenney, as Commander of the Allied Forces, received intelligence reports that part of the Japanese Navy was about to sail from Rabaul, in the island of New Britain, to Lae, in New Guinea.
- His mission was to intercept the convoy and to bomb it to the maximum extent possible.

- To complicate matters, Kenney also received weather reports of rain and poor visibility in the area north of New Britain, but good weather and visibility in the south of the island.
- The Japanese commander had two possible courses of action: either (a) he could sail his convoy north of the island, or (b) he could go south of the island. Any of these routes would take three days to sail.
- General Kenney also had two possible courses of action: either (a) he could concentrate most of his reconnaissance aircrafts to search the northern route and send a few to search the southern route, or (b) vice versa.
- Suppose the US focused north, and Japanese also went north. Because of poor visibility, the convoy wouldn't be discovered until the second day, allowing for two days of bombing.

- Suppose the US focused north, but Japanese went south. Because of limited reconnaissance south of the island, the convoy could be missed during the first day, allowing once again for two days of bombing.
- Suppose the US focused south, but Japanese went north. Considering the poor visibility north of the island, plus limited reconnaissance, the convoy would be missed for two days, allowing for just one day of bombing.
- Suppose the US focused south, and Japanese also went south. In this case, having the majority of airplanes in the area and having good visibility, General Kenney could hope for three days of bombing.
- What should General Kinney have done?

Two-Player Games with No Nash Equilibrium

Rock Paper Scissors

- Two players simultaneously choose ("throw") one of three hand signals: Rock is a fist, Paper is a flat palm, and Scissors is a V formed by the index and middle finger pointing to the opponent.
- If the two players make the same choice, the result is a tie. If the two make different choices, then Rock wins against (breaks) Scissors, Scissors wins against (cuts) Paper, and Paper wins against (covers) Rock.
- What is each player's optimal strategy?

Tennis Passing Shot

- Imagine a tennis match between top-seed women's players like Simona Halep and Caroline Wozniacki.
- Halep is at the net and has just volleyed a ball to Wozniacki on the baseline, and Wozniacki is about to attempt a "passing shot". She can try to send the ball down the line (DL) or cross court (CC). Halep must likewise prepare to cover one side or the other.
- Rewards are defined as the percentage of times a player wins the point in any combination of passing shot and covering play. Given that a down-the-line shot is stronger and shorter than a cross-court shot, and that Wozniacki is more likely to win the point when Halep moves to cover the wrong side of the court, we can work out a reasonable set of payoffs.

		Halep	
		DL	**CC**
	DL	50%	80%
Wozniacki	**CC**	90%	20%

- What is each player's optimal strategy?

Two-Player Games with Multiple Nash Equilibriums

Chicken

- Imagine yourself as an American male teenager in the 1950s. You live in a small town named Middle-of-Nowhere. It is a late Saturday evening. You are with a group of friends, playing games of rivalry to decide who is the alpha male in the group. Tonight's contest starts with the game of chicken.
- Two of you get in your cars on opposite ends of Main Street. You race directly towards each other, in what appears to be a head-on collision. The first one to swerve is the loser, or "chicken". The other person wins. You want to win.
- This is a dangerous game. If both of you attempt to win, both will end up in the hospital – or worse.
- Assume the following reward system, where reward is combination of prestige among peers and physical well-being: (a) if both swerve, then both get 0 points (b) if one swerves and the other drives straight, then the one who swerved gets -20 points, and the one who drove straight got 20 points (c) if both drive straight: both get -100 points
- What is each teenager's optimal strategy?

Battle of the Spouses

- A husband and wife have plans to go to the theater to watch a movie.
- The husband likes to see fighting movies that do not tax the brain – like *Gladiator*. The wife likes to see cerebral movies with a bit of weeping –like *Pride and Prejudice*.
- Despite their differences, they would rather give each other company in a movie they hate than go to separate movies and be without their spouse.
- Assume the following reward system: (a) if both watch a fighting movie, the husband gets 3 points and the wife gets 1 point (b) if both watch a cerebral movie, the wife gets 3 points and the husband gets 1 point (c) if both go to their own separate movies, then both get 0 points
- What is each spouse's optimal strategy?

N-Player Games

Tragedy of the Commons

- In the late 1800s, cattle herders shared a common parcel of land ("the commons") on which they were could let their cows graze.
- If a herder put more than his fair share of cattle on the common, then he would benefit from free grazing
- But the overgrazing would eventually (over several months) cause the commons to become grassless, and the pain would be shared by all the herders, including the ones who had kept to their fair share
- Today we have the same problem in different guises: our commons are rivers, oceans, and the atmosphere.
- Formulate this as a simple game, and explain why the optimal strategy is indeed a "tragedy"

Changing the Game!

Chicken again!

- <Idea of Commitment>

Tragedy of Commons again!

- <Idea of Mechanism Design>

Applications to Economics

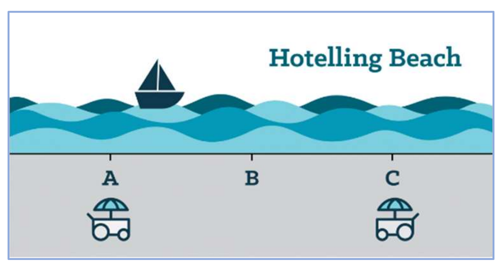
Competition

- Consider a small college town with a population of dedicated pizza eaters but able to accommodate only two pizza shops, Donna's Deep Dish and Pierce's Pizza Pies
- The <u>cost</u> of making a pizza is $5 for both shops
- The <u>price</u> of selling a pizza is up to each shop. To keep things simple, assume that there are only three options for price: (a) low price $10 (b) medium price $15 or (c) high price $17
- Each store has a loyal customer base who will buy 3000 pizzas per week, no matter what price is charged in either store.
- In addition, there is a floating customer base who will buy 4000 pizzas per week from whichever store has the lower price; if both stores have the same price, then the demand is split equally between them
- How should the two pizza shops set their respective prices so as to maximize their respective profit?

Applications to Politics

Presidential Elections

- Assume there are two presidential candidates. Each of them state their *positions*, i.e. what they will do if they are elected. Voters pick the candidate whose position they like more.
- How should politicians should pick their positions so as to attract the greatest number of voters? This was the problem studied by **Hotelling**.
- For simplicity, Hotelling assumed that political positions can be summarized on a left-right spectrum. For example, in the US, Democrats typically take "leftist" position, and Republicans take a "rightist" position.
- Since politics may be boring to you, let's make an analogy with something more interesting: ice cream at the beach!
- There are two ice cream sellers, Donald and Elizabeth. Each wants to set up an ice cream stand on a beach.



- This beach has 100 sunbathers. Every so often, a sunbather stands up, looks around, and walks to the ice cream stand closer to him/her.
- Where should Donald and Elizabeth position their stands so as to attract the most number of customers?
- (Note the analogy: ice cream sellers = presidential candidates; sunbathers = voters; and position of the ice cream = political position of candidates)