

Applications of Catalan Numbers

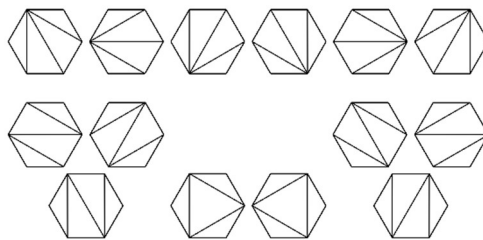
Compiled by Raghu Subramanian – for San Jose Math Circle

Problem 1

- There are $2n$ people in a movie queue. Of them, n have exactly one \$10 bill, and the other n have exactly one \$20 bill. The movie ticket costs \$10, and the box-office starts off with no change. We know that there are $\binom{2n}{n}$ ways of arranging the n customers-with-\$10 and the n customers-with-\$20 in a queue. But in how many ways can they be arranged in a *hiccup-less* queue -- defined as a queue where no \$20-customer will have to step aside and wait for change.

Problem 2 [by Euler]

- How many ways are there of triangulating a convex polygon with $(n + 2)$ sides?
- As an example, a convex hexagon can be triangulated in the following ways:



Problem 3

- How many “mountain ranges” can you draw with $2n$ strokes?
- As an example, here are the mountain ranges with 6 strokes



- Note that our definition of a mountain range (a) must end at the level it started, and (b) must never dip below the level it started

Problem 4

- How many “sensible” ways are there to type a string of n left and n right parentheses?
- For example: $((()))$ and $()(())$ are sensible; but $))(($ and $()(())$ are nonsensical

Problem 5

- Given $2n$ points on a circle, how many ways are there to pair them using n chords in such a way that no two chords intersect
 - Tom Davis likes to visualize this as $2n$ children sitting around a circular table, where all of them are simultaneously shaking hands with some other person in such a way that none of the arms cross.

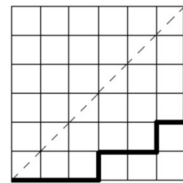
Problem 6

- How many “trees” can you draw with n binary forks?
- For example, you can draw 5 trees with 3 binary forks (indicated by dark dots below)



Problem 7

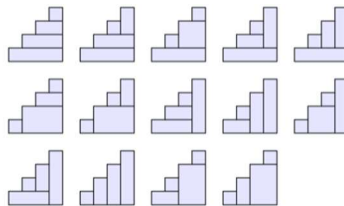
- A Manhattan-like city has a grid of n N-S streets and n E-W streets. You want to walk from the SW corner to the diagonally opposite NE corner by zigzagging up and to the right. It is well known that you have $\binom{2n}{n}$ paths available.



- But suppose the portion of the city that is NW of the diagonal dotted line is unsafe; so you want to choose a path that stays SE of the diagonal dotted line (touching the diagonal is okay). Now how many paths are available to you?

Problem 8

- How many ways are there to build an n -step staircase with n rectangular slabs
- For instance, here are all the ways to build a 4-step staircase with 4 rectangular slabs



Problem 9 [from Chemical Engineering]

- *Fractional distillation* is a versatile technique to separate a mixture into its components -- e.g. to separate crude oil into gasoline, naphtha, paraffin, kerosene, diesel oil, lubricating oils, wax etc. The basic step of fractional distillation is to heat the mixture to some temperature where a subset of components will vaporize but the remaining subset will stay liquid. How many “separation schemes” are there for a mixture of $n + 1$ components?