Slicing and Dicing Problems

Compiled by Raghu Subramanian – for San Jose Math Circle

Warmups

Exercise 1

- You are given a cube of cheese 4cm x 4cm x 4cm, and sharp-edged knife. You are to cut the cheese into 64 little cubelets, each 1cm x 1cm x 1cm. What is the minimum number of cuts you need?
- Suppose you can re-position the pieces between cuts – e.g. you can form a stack a few slices and cut them all together. Now what is the minimum number of cuts you need?
- Repeat both the above problems with a 3cm x 3cm x 3cm. How come you can't reduce the number of cuts?

Exercise 2

- What is the maximum number of pieces in which a circular birthday cake may be divided with three cuts, if all the pieces must be the same size? (Otherwise you will have some unhappy children to deal with!)

Exercise 3

- There is a simple procedure by which two children can divide a cake so that each is satisfied that he or she has at least half: one cuts and the other chooses. Can you extend this idea to three children?
- Devise a general procedure so that n persons can cut a cake into n portions in such a way that everyone is satisfied that he or she has at least 1/n of the cake.

Exercise 4

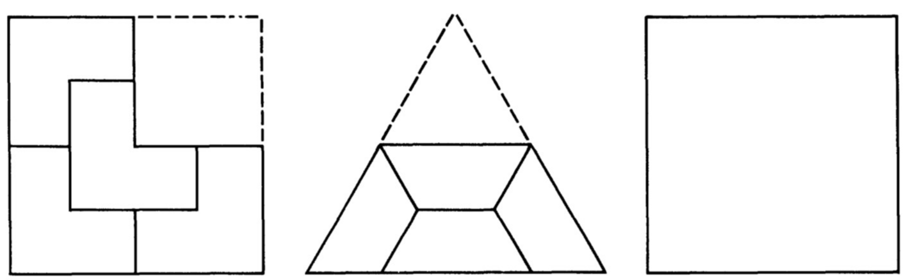
- There is a cubical piece of jello, which has a smaller cubical cavity (air bubble) inside it. If you peer carefully, you can see the cavity: it is arbitrarily oriented, i.e. its faces are not parallel to the faces of the jello. With one plane slice of a knife, cut the jello into two pieces that are equal in (jello) volume.
- Now there are two cubical cavities of different sizes and orientations. How can you extend your solution?

Exercise 5

- From a 2x2 square, a 1x1 quadrant is removed from the north-east corner, leaving an L-shaped figure. Bisect the area of the L-shaped figure using a straight cut that passes through the north-west corner of the original 2x2 square.

Exercise 6

- A farmer has a plot of land that looks like one-fourth a square cut from its corner (see figure below). He has four children, among whom he wishes to divide the land – not only equally by area, but also congruent in shape. It is possible to do so, as shown in the figure.
- Another farmer has a plot of land that looks like one-fourth an equilateral triangle cut from its corner (see figure below). He has four children, among whom he wishes to divide the land – not only equally by area, but also congruent in shape. It is possible to do so, as shown in the figure.
- A third farmer has a plot of land that a full square – nothing removed. But he has _five_ children. Can he divide the land among his five children so that each gets the same area and congruent shape?



Exercise 7

- You have a piece of paper in the shape of an equilateral triangle. With a pair of scissors, you want to snip off portions from the three corners so as to leave a regular hexagon. How do you figure out where to cut using a straight edge (i.e. unmarked ruler) and compass?
- Do the same with a square, so as to leave a regular octagon. Once again, all you can use is a straight edge and compass.

Exercise 8

- What is the minimum number of cuts you must make to a 12 feet x 12 feet carpet so that it fits a 9 feet x 16 feet room?
- What if zig-zag cuts are allowed? Now what is the minimum number of cuts you need?

Intermediate Level

Exercise 9

- Each of the vertices of the base of a triangle is connected by straight lines to n points on the side opposite to it. Into how many parts do these 2n lines divide the triangle?
- Now include the top vertex as well, with n points on base. Into how many parts do these 3n lines divide the triangle?

Exercise 10

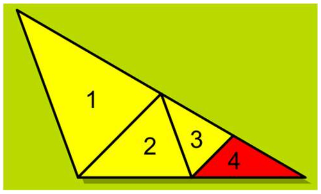
- Place n points anywhere you like on the circumference of a circle. Connect each point to every other point. What is the maximum number of regions that you can form this way?

Exercise 11

- Cut a square A into four rectangular pieces such that two of these pieces can be reassembled to form a smaller square B, and the other two pieces can be reassembled to form an *even* smaller square C. (That is to say, B and C must be unequal sizes.)

Exercise 12

- An acute triangle is a triangle with all three angles acute. An obtuse triangle is a triangle with one obtuse angle. Given an obtuse triangle, is it possible to cut it into acute triangles? The figure below shows a failed attempt, because triangles 1, 2, and 3 are acute, but 4 is not. If it is impossible, give a proof of impossibility. If it is possible, what is the smallest number of acute triangles into which you can guarantee to cut any obtuse triangle



Exercise 13

- Can you cut a hole through a cube that allows a larger cube to pass through?

Exercise 14

A whiz computer scientist in the bay area was having coffee and a bagel. Before she dropped a sugar cube into her cup, she placed the cube on the table and thought:

- “If I pass a horizontal plane through the cube's center, the cross section will of course be a square. If I pass it vertically through the center and four corners of the cube, the cross section will be an oblong rectangle. Now suppose I cut the cube this way with the plane...” To her surprise, her mental image of the cross section was a regular hexagon!

- How was the slice made? If the cube's side is half an inch, what is the side of the hexagon?

After dropping the cube into the coffee, she turned her attention to the unsliced bagel lying flat on a plate.

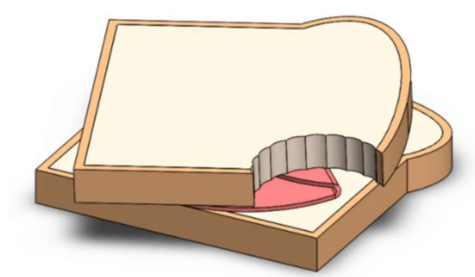
- “If I pass a plane horizontally through the center, the cross section will be two concentric circles. If I pass the plane vertically through the center, the section will be two circles separated by the width of the hole. But if I turn the plane so...” She whistled with astonishment. The section consisted of two perfect circles that intersected!
- How was this slice made? If the doughnut is a perfect torus, three inches in outside diameter and with a hole one inch across, what are the diameters of the intersecting circles?

Exercise 15

- A thousand poppy seeds are scattered inside a muffin. Is it always possible to slice the muffin with a plane so that exactly 500 poppy seeds lie on each side of the plane? The answer is yes; prove it!

Exercise 16

- Suppose you are given a sloppily constructed ham sandwich: two slices are different sizes and shapes – in fact someone has taken a bite out of one slice -- and the slice of ham is haphazardly folded between the slices (see figure below). The famous **ham sandwich theorem** says that you can make a single plane cut with a knife that will simultaneously bisect the volume of all three items.
- In fact, this theorem applies to any three bodies in three-dimensional space. They don't have to be flattish in shape, and don’t even have to be near each other!



Advanced Level

Exercise 17

- What is the greatest number of parts into which a plane can be divided by n straight lines?

Exercise 18

- What is the greatest number of parts into which three-dimensional space can be divided by n planes

Exercise 19

- Suppose in exercise 17, you replace “straight lines” with circles. Now how many regions?
- Suppose in exercise 18, you replace “planes” with spheres. Now how many regions?

Exercise 20

- On a circular sheet of paper, $2n$ points are marked on the circumference -- with different colors so that you can tell them apart. You cut this piece of paper along a chord joining two adjacent points. Then, you cut the remaining piece of paper along a chord joining two adjacent points. And so on, until you have made n cuts. Note that a pair of points that were originally non-adjacent may become adjacent along the way (because the intervening points have been snipped off): it is permitted to cut along the chord joining such a pair. How many ways are there of cutting up the circular sheet of paper?