Probability, Markov Processes, and an old Casino Game

San Jose Math Circle **Ted Alper**
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1 Warm-up Problems – or are they?

(2006 AMC 10) A player pays \$5 to play a game. A die is rolled. If the number on the die is odd, the game is lost. If the number on the die is even, the die is rolled again. In this case the player wins if the second number matches the first and loses otherwise. How much should the player win if the game is fair? (In a fair game the probability of winning times the amount won is what the player should pay.)

Problem 1 (Mathcounts) Aiden, Brayden, and Claydon take turns flipping a coin in alphabetical order. The first to flip heads wins (if none of them flip heads, the game restarts from Aiden). What is the probability that Aiden wins?

Problem 2 (2001 AMC 10) A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

Problem 3 Two cows stand on opposite faces of a cube and take turns moving to a random adjacent face. A cow moving into the same face as another cow knocks the other cow over. What is the probability that the first cow to move eventually knocks over the second cow?

Problem 4 Suppose that the two cows decide to make their game more exciting and play it on the vertices of the cube, starting on opposite vertices. What is the probability that the first cow to move eventually knocks over the second cow?

Problem 5 (2014 AMC10B #25 & AMC12B #22) In a small pond there are eleven lily pads in a row labeled 0 through 10. A frog is sitting on pad 1. When the frog is on pad N, $0 < N < 10$, it will jump to pad $N-1$ with probability $\frac{N}{10}$ and to pad $N+1$ with probability $1-\frac{N}{10}$. Each jump is independent of the previous jumps. If the frog reaches pad 0 it will be eaten by a patiently waiting snake. If the frog reaches pad 10 it will exit the pond, never to return. What is the probability that the frog will escape being eaten by the snake?

(A) $\frac{32}{79}$ **(B)** $\frac{161}{384}$ **(C)** $\frac{63}{146}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{1}{2}$

Problem 6 (1995 AIME) If you flip a fair coin repeatedly, what is the probability you see a run of 5 consecutive heads before seeing a run of 2 consecutive tails?

Problem 7 In a gambling game, you start with \$ 1, and every time you play, you are equally likely to win or lose \$ 1. What is the probability that you reach \$ 10 before you go broke?

Problem 8 What are your expected winnings in this game, assuming you decide to play until you reach \$ 10 or you go broke?

Problem 9 Suppose you are quite good at the gambling game, and you win \$ 1 with a probability of 2 3 (and lose \$ 1 otherwise). You still start with \$ 1, and keep playing as long as you can. What is the probability that you never run out of money?

Problem 10 (2009 AIME) Dave rolls a fair six-sided die until a six appears for the first time. Independently, Linda also rolls a fair six-sided die until a six appears for the first time. What is the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die?

Problem 11 We flip a fair coin repeatedly. *Before* each flip, you can decide whether you want the flip to be worth 1 point or 2 points. If it comes up heads you get the point or points; if it comes up tails, I get the point or points. The first person to get at least 100 points wins. (if that's too high to analyze, we could make the cutoff 5 points or even 3 points). Is this game fair, or do you have an advantage?

Problem 12 (From Vladimir Arnold's Trivium but a little off-topic): One player conceals a 10 or 20 kopeck coin, and the other guesses its value. if he is right, he gets the coin, if wrong, he pays 15 kopecks. Is this a fair game? What are the optimal strategies for both players? (what kinds of strategies are allowed?)

2 Some Data Gathering

We need some sequences of random coin tosses to analyze. We could use http://random.org (and maybe we will), but I want you to generate them yourselves. It's actually a little faster and neater to roll dice rather than coins, so I'm giving you 10-sided dice to roll. Record odd numbers as "H" (for heads) and even numbers as "T" (for tails). Roll repeatedly, writing down the sequence of "T" and "H" you get until your sequence includes at least one occurrence of "HHT" and at least one occurence of "THH" and at least one occurence of "TT". This might happen fairly quickly or it might take 20 rolls or more. Once that happens, that sequence is done, but you can start another sequence, again rolling all three of those sequences appear somewhere. Make at least five such sequences.

	First Occurence of					
Your Sequence	H	T	HH	TH	HHT	THH

For each sequence, we want the location of the following:

- first occurence of "H" and first occurence of "T"
- which occured first "H" or "T"
- first occurence of "HH" and first occurence of "TH" (use the position of the first coin)

- which occured first, "HH" or "TH" ?
- first occurence of "HHT" and first occurence of "THH"
- which occured first, "HH" or "TH" ?

For example, if I rolled "HTHTTHHT", I'd say the first occurence of H happenned in position 1, the first occurence of T happened in position 2, the first occurence of HH happened in position 6, the first occurence of "TH" happened in position 2, the first occurence of HHT happened in position 6 and the first occurence of THH happened in position 5. H occured before T, TH occured before HH and THH occured before HHT.

Does that make sense? We're going to collect answers for all of these and calculate the average values for these, but – before we do – can we make some guesses as to what we'd expect?

3 Different Topic: Chuck-a-Luck

Chuck-a-Luck is a casino game, though not as popular now as it was a hundred years ago. It is conventionally played with 3 six-sided dice, but we can try it with 2 four-sided dice or 5 ten-sided dice (or more generally, n $2n$-side dice.

In the game, you Make a \$1 bet and pick one number between 1 and $2n$. then roll n $2n$-sided dice. If your number comes up on exactly k of the dice, you win \$ k (where $N = 1, 2, 3, 4, n$, of course But if your number doesn't come up on any of the dice, you lose your \$1 bet.

So for example, with four-sided dice: Make a \$1 bet and pick one number from $\{1, 2, 3, 4\}$ and roll ***two*** 4-sided dice. If your number comes up on one of the dice, you win \$ 1 if it comes up on **both** dice, you win \$2! But if it doesn't come up at all, you lose your bet.

With six-sided dice, it's the same idea: Make a \$1 bet and pick one number from $\{1, 2, 3, 4, 5, 6\}$ and roll ***three*** 6-sided dice. If your number comes up on one die, you win \$1, if it comes up on two of the dice, you win \$2 and if it comes up on all three dice, you win \$3. But if it doesn't come up at all, you lose your bet.

and with ten-sided dice, you roll five dice, winning between \$1 and \$5 if your number comes up, or losing your bet if it doesn't come up at all.

Problem 13 Pick one or more of the games. Play it at least 32 times and keep a record of how well you do – how many times you win and how many times you lose. Then try to answer these questions:

(i) is this a fair game? If not, how could you change the payout to make it fair?

(ii) if you play it 1000 times, how many times do you expect to win? How many times do you expect to lose. If you bet \$ 1 every time, how much money do you expect to win or lose?

(iii) if you start with \$1 and play until you either win \$10 or go broke, what is the probability you go broke?

(iv) if you start with \$2 and play until you either win \$10 or go broke, what is the probability you go broke?

BETTING LAYOUT for 2 four-sided dice

1	2	3	4

BETTING LAYOUT for three six-sided dice

1	2	3
4	5	6

BETTING LAYOUT for 5 ten-sided dice

1	2	3	4	5
6	7	8	9	0