Monthly Contest #2: Upper Division

Due: February 1, 2019

San Jose Math Circle

January 2019

Instructions: This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. It would be preferred if you used 8.5 by 11 printer paper to write your solutions on and leave a small margin around the edges. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problems solution (not the total pages of all the solutions). DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Leave answers in exact form unless otherwise specified. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems. You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may not view any book or website (including forums) unless otherwise stated in the problem.

Problems

Problem 1. In a certain country 10% of the employees get 90% of the total salary paid in this country. Suppose the country is divided into several regions. Is it possible that in every region the total salary of any 10% of the employees is no greater than 11% of the total salary paid in this region?

Problem 2. Let n be a positive integer. Prove that there exist n positive integers, none of which is prime.

Problem 3. Let n be integer. There are n students with water guns standing on a field. Each student then shoots the student closest to them. For which n are all students drenched?

Problem 4. Let a, b, c be positive integers satisfying the following two equations:

$$a^4 + a^2b^2 + b^4 = 9633$$
$$2a^2 + a^2b^2 + 2b^2 + c^5 = 3605.$$

Find all possible triples (a, b, c).

Problem 5. Let n be an odd positive integer, and let $x_1, x_2, \cdots, x_n$ be non-negative real numbers. Show that

$$\min_{i=1,\ldots,n}(x_i^2 + x_{i+1}^2) \leq \max_{j=1,\ldots,n}(2x_j x_{j+1})$$

where $x_{n+1} = x_1$.