San Jose Math Circle 2018
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Permutations and puzzles

Terms and techniques mentioned/used: Bijective function, permutation, transposition, cycle, floor function, parity, Hamiltonian path, Hamiltonian circuit

1. Define permutation.

2. **Problem 1** (Khang N Thanh)
Find the number of permutations f on $A = \{1, 2, 3, ...32\}$ satisfying the following property. If m divides n then $f(m)$ divides $f(n)$.
I.e. $m|n => f(m)|f(n)$.

3. Define cycles, transpositions (swaps).

4. **Problem 2 (Boxes)** (Asana folklore)
There are 100 prisoners numbered 1 through 100, and each prisoner knows his own number. There is a room with 100 boxes also numbered 1 through 100 on the cover. The warden has written the numbers 1 through 100 on 100 index cards, and randomly placed them inside the boxes. Each prisoner then enters the room one at a time and can open one of the boxes and see if the index card in it has his own number (he must leave the index card inside the box). He can then close that box and look inside another box, and continue this process up to 50 times or until he sees an index card with his own number. He then reports the box number to the warden and leaves the room as it was before he entered the room. The warden lets all of the prisoners go, if they all report the correct number. If any one of the prisoners is unable to view his own index card, none of the prisoners are let go. Knowing this procedure, the prisoners manage to work out a strategy so that they have a non-miniscule chance of success. This strategy works for not only 100 prisoners, but for any number of prisoners, where each prisoner can view half the total boxes. What is the strategy?

5. Proof for parity of A_n

6. **Problem 3 (Mittens)** (Asana folklore)
There are n prisoners. The warden writes down some rational number of his choosing on each of their foreheads. Each prisoner can see the numbers on everyone else's foreheads. The prisoners are then separated and taken into private cells, where they are given one black mitten, and one white mitten each. Each prisoner has to put on one of the mittens on his left hand, and the other on his right. The warden then takes them out of the cells, and now lines them up side by side in the order of the numbers on their foreheads, from lowest to greatest. The prisoners then have to reach out to the side and hold hands with the prisoner(s) next to them. The warden then looks at each of the handholds, and checks to see whether the mittens match in color. If all of the mittens match, then the warden sets everyone free. Knowing this procedure, the prisoners

have come up with a strategy such that they are guaranteed to go free. What is their strategy?

7. **15-puzzle proof** (Aron Archer)

8. Enrichment **Problem 4 (15 puzzle proof modification)**

a) For an arbitrary Hamiltonian path, would you always get 9 non-identity permutations associated with it?
b) Choose a Hamiltonian path and write the non-identity permutations for this path.
c) How about a Hamiltonian circuit? Is there any advantage? Create a Hamiltonian circuit and write the non-identity permutations for this circuit.

Two fun problems to work on

9. **Prablem 5 (Hats)** (Asana folklore)

There are n prisoners. The warden lines them up so that each prisoner can only see all the prisoners in front of him. The warden places either a red hat or a blue hat on each prisoner's head. Now each can see the colors of the hats in front of him but not that of his own or of the hats behind him. The warden then starts from the back and asks each prisoner to loudly guess the color of his own hat, and if he guesses correctly, the prisoner is allowed to go free. Having been told the procedure beforehand, the prisoners have worked out a strategy such that every prisoner except the prisoner at the back is guaranteed to go free. What is the strategy?

10. **Problem 6 (number on forehead)**

There are n prisoners. The warden chooses one number from 1 through n for each prisoner and puts it on that prisoner's forehead. He may repeat the same number for multiple prisoners. He then has each of them write down a guess of the number on their own forehead and looks through the guesses. If any one of them guesses his own number correctly, then all are allowed to go free. Knowing this procedure, the prisoners work out a strategy in advance to ensure they are guaranteed to go free. What is the strategy?