Monthly Contest 1
Due November 3, 2017

Instructions

This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. It would be preferred if you used 8.5 by 11 printer paper to write your solutions on and leave a small margin around the edges. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problems solution (not the total pages of all the solutions).

DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Leave answers in exact form unless otherwise specified. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems. You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. **You may not view any book or website (including forums) unless otherwise stated in the problem.**

Problems

1. A spider owns eight distinct socks and eight distinct shoes. For any leg, the spider must put on its sock before its shoe. Find the number of orders in which the spider can put on its socks and shoes. You may leave your answer in unexpanded form.

2. Mathopolis has a new road with 5 blocks marked for construction. A house, restaurant, and hotel each takes up 1 block of land, while an apartment complex and a school each takes up 2 blocks. How many ways are there to allocate these 5 blocks if these 5 buildings are the only options for construction?

3. Prove that for all nonnegative integers n,

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} = 4^n$$

4. $2n$ dots are placed around the outside of a circle. n of them are colored red and the remaining n are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots passed, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the correct point.

5. For $0^\circ \leq x \leq 45^\circ$, the values of $\cos(x)$, $\sin(x)$, and $\cos(2x)$ form an arithmetic sequence (but not necessarily in that order). What is the common difference of this arithmetic sequence?