

Monthly Contest 5
Due April 6, 2018

Instructions

This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. It would be preferred if you used 8.5 by 11 printer paper to write your solutions on and leave a small margin around the edges. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problems solution (not the total pages of all the solutions).

DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Leave answers in exact form unless otherwise specified. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems. You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. **You may not view any book or website (including forums) unless otherwise stated in the problem.**

Problems

1. Let a, b and c be real numbers such that $\frac{ab}{a+b} = \frac{1}{3}$, $\frac{bc}{b+c} = \frac{1}{4}$, and $\frac{ca}{c+a} = \frac{1}{5}$. Find the value of $\frac{24abc}{ab+bc+ca}$.
2. Let x and y be real numbers satisfying the inequality

$$5x^2 + y^2 - 4xy + 24 \leq 10x - 1.$$

Find the value of $x^2 + y^2$.

3. Two ducks, Wat and Q, are taking a math test with 1022 other ducklings. The test has 30 questions, and the n th question is worth n points. The ducks work independently on the test. Wat gets the n th problem

correct with probability $\frac{1}{n^2}$ while Q gets the n th problem correct with probability $\frac{1}{n+1}$. Unfortunately, the remaining ducklings each answer all 30 questions incorrectly.

Just before turning in their test, the ducks and ducklings decide to share answers! On any question which Wat and Q have the same answer, the ducklings change their answers to agree with them. After this process, what is the expected value of the sum of all 1024 scores?

4. What is the greatest number of sections into which a plane can be divided by n circles?
5. Using only the AM-GM Inequality, prove that for $x, y, z \geq 0$,

$$x^5yz + xy^5z + xyz^5 \geq x^3y^2z^2 + x^2y^3z^2 + x^2y^2z^3.$$