2016-17 Monthly Contest 2 Solutions

Note: The following solutions are not the only solutions possible. We encourage you to seek other solutions, and perhaps yours will be more elegant than ours!

1. Given a regular convex pentagon ABCDE and 6 colors,

   (a) Find the number of all possible colorings of the vertices such that no edges connect vertices of the same color.

   (b) Find the number of all possible colorings of the vertices such that no diagonals connect vertices of the same color (edges can connect vertices of the same color).

   Solution:

   (a) We can color A in 6 ways. After coloring A, there are 5 ways to color B, because A and B cannot be the same color. Thus, there are $6 \cdot 5 = 30$ ways to color A and B.

   To deal with the remaining vertices, we can do casework on the relationship between A and C.

   Case 1: A and C are the same color.
   Since we have already chosen A's color, C only has 1 color option as well. Since C and D are adjacent, D cannot have the same color as C, so D has 5 remaining coloring choices. This also means that D and A have different colors. This leaves E with 4 remaining colors, since E borders A and D. Thus, there are $1 \cdot 5 \cdot 4 = 20$ color combinations for C, D, and E in Case 1.

   Case 2: A and D are the same color.
   This case follows the same logic as case 1. Since we have already chosen A's color, D only has 1 color option as well. Since C and D are adjacent, C cannot have the same color as D or B, so D has 4 remaining coloring choices. Because A and D are the same color, E can be any of the 5 other colors. Thus, there are $1 \cdot 5 \cdot 4 = 20$ ways to color points C, D, and E in Case 2.

   Case 3: A, C, and D are different colors.
   Because point C is different from A and is adjacent to B, there

are 4 colors left for B. This is the same case for D. Because it is different from A and adjacent to C, there are also 4 ways to color it. Because E is adjacent to both A and D (which are different colors), there are 4 color choices for E. This gives us $4 \cdot 4 \cdot 4 = 64$ ways to color points C, D, and E.

There are a total of $30 \cdot (20 + 20 + 64) = \boxed{3120}$ combinations to color the vertices of ABCDE.

(b) One thing to note about this new figure, is that these graphs are isomorphic. This means that they have the same number of vertices, edges, and edge connectivity. You can rearrange the points on this graph to get the same pentagon as before but with the points relabeled. You end up with pentagon ADBEC, whose diagonals are AB, BC, CD, DE, and EA. Thus, this problem has been reduced to part a) and the answer is also $\boxed{3120}$.

2. Find the remainder when $e^{2016(\ln 7)+7(\ln 2016)}$ is divided by 11.

Solution: $e^{2016(\ln 7)+7(\ln 2016)}$ can be rewritten as

$$\begin{aligned} e^{2016(\ln 7)+7(\ln 2016)} &= e^{2016(\ln 7)} \cdot e^{7(\ln 2016)} \\ &= (e^{\ln 7})^{2016} \cdot (e^{\ln 2016})^7 \\ &= 7^{2016} \cdot 2016^7 \end{aligned}$$

So we want to solve for the value of $7^{2016} \cdot 2016^7 \pmod{11}$. We observe that $7^5 \equiv 10 \equiv -1 \pmod{11}$, so 7^{2016} can be rewritten as

$$7^{2016} = (7^5)^{403} * 7 \equiv (-1)^{403} \cdot 7 \equiv -7 \equiv 4 \pmod{11}.$$

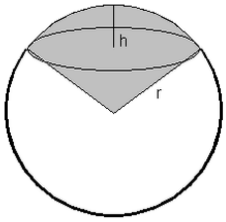
Since $2016 \equiv 3 \pmod{11}$ and $3^3 = 27 \equiv 5 \pmod{11}$,

$$2016^7 \equiv 3^7 \equiv (3^3)^2 \cdot 3 \equiv 25 \cdot 3 \equiv 75 \equiv 9 \pmod{11}.$$

Thus, $e^{2016(\ln 7)+7(\ln 2016)} \pmod{11} = 4 \cdot 9 = 36 \equiv \boxed{3} \pmod{11}$.

3. Find the volume of the solid which is the intersection of two unit spheres with their centers one unit apart. (Hint: The volume of a spherical sector is $\dfrac{2\pi r^2 h}{3}$, where r and h are as labeled in the figure below.)

Solution: Note that the volume of the intersection is 2·(Spherical sector− cone).



The volume of a spherical sector is:

$$\frac{2\pi r^2 h}{3} = \frac{2\pi(1^2)(\frac{1}{2})}{3} = \frac{\pi}{3}.$$

We can find that the radius of the cone base is $\frac{\sqrt{3}}{2}$ by using the Pythagorean theorem. From this, we find that the volume of the cone is:

$$\frac{\pi r^2 h}{3} = \frac{\pi(\frac{\sqrt{3}}{2})^2(\frac{1}{2})}{3} = \frac{\pi}{8}$$

Thus the area of the intersection is $2(\frac{\pi}{3} - \frac{\pi}{8}) = \boxed{\frac{5\pi}{12}}$.

4. Find all natural numbers a, b, c such that a!+b!=c!. Prove that you have found all of them.

   Solution: The following solution applies for the definition of natural numbers that includes 0 as a natural number. Otherwise, the only solution is (1,1,2). Both sets of answers were given full credit.

   We can immediately conclude the $c > a$ and $c > b$, because otherwise the left-hand side would be larger than the right-hand side of the equation. As a result, $(c-1) \geq a$ and $(c-1) \geq b$.

   We can also conclude that $c < 3$ from a short contradiction argument. Suppose $c \geq 3$. Then, multiplying both sides of the inequality by $(c-1)!$,

$$\begin{aligned} c! &\geq 3(c-1)! \\ &= (c-1)! + (c-1)! + (c-1)! \\ &= a! + b! + (c-1)! \\ &> a! + b!. \end{aligned}$$

   Since $c < 3$, we only need to check c=0,1, and 2. There are no solutions for c=0 and c=1. For c=2, the only solutions are $\boxed{(0,0,2),(0,1,2),(1,0,2),(1,1,2)}$.

5. Nina chooses a coin from one hundred pennies and promptly tosses six heads in a row. However, she soon discovers that four of the 100 pennies were two-headed. Find the probability that Nina chose one of the two-headed pennies given that she flipped six heads in a row.

   Solution: This is a scenario of conditional probability. This can be solved through either casework or Bayesian probability, which if you don't know, you should definitely check out!

   There are 2 cases for Nina to have flipped 6 heads in a row: the case where Nina chose the two-headed penny and the case where she didn't.

   Case 1: Nina flipped six heads using the two-headed penny. The probability Nina picked a two-headed penny is $\frac{4}{100} = \frac{1}{25}$. The probability that Nina flipped six heads is $(1)^6$, because the only option is to flip a head. Thus the probability for Case 1 is

$$\frac{1}{25} \cdot (1)^6 = \frac{1}{25}.$$

   Case 2: Nina flipped six heads using the regular penny. The probability Nina picked a regular penny is $\frac{96}{100} = \frac{24}{25}$. The probability that Nina flipped six heads is $(\frac{1}{2})^6$, because now Nina has two options per flip. The probability for Case 2 is

$$\frac{24}{25} \cdot (\frac{1}{2})^6 = \frac{3}{200}.$$

   The probability that Nina chose one of the two-headed pennies given that she flipped six heads in a row is the probability that Case 1 occurred out of all possible cases (Case 1 and Case 2). The final probability is

$$\frac{\frac{1}{25}}{\frac{1}{25} + \frac{3}{200}} = \boxed{\frac{8}{11}}.$$