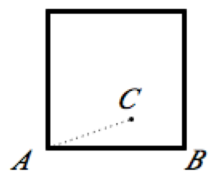
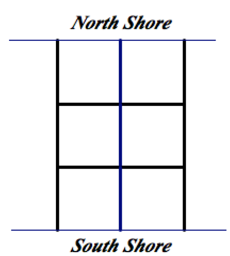
San Jose Math Circle — Problems for 11/2/16

1 *Planes.* A group of airplanes is based on a small island. The tank of each plane holds just enough fuel to take it halfway around the world. Any desired amount of fuel can be transferred from the tank of one plane to the tank of another while the planes are in flight. The only source of fuel is on the island, and we assume that there is no time lost in refueling either in the air or on the ground. What is the smallest number of planes that will ensure the flight of one plane around the world on a great circle, assuming that the planes have the same constant ground speed and rate of fuel consumption and that all planes return safely to the island base?

2 *Billiard ball.* A mathematical billiard ball (i.e., a point with zero radius) is shot from corner A of the square below as shown on a line with a slope of $13/40$. How many times will it bounce off a wall of the square before it returns to a corner, and which corner will it return to?

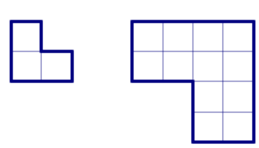


3 *Bridges.* A system of 13 bridges, shown below, connects the north shore of a river to the south shore. For each bridge, there is a 50% probability that a protest march will block traffic across that bridge, and these probabilities are independent (imagine that each bridge flips a coin). What is the probability that it is possible to cross from one shore to the other?



4 *Cubic Coloring.* Is it possible to color the faces of 27 identical $1 \times 1 \times 1$ cubes, using the colors red, white, and blue, so that one can arrange them to form a $3 \times 3 \times 3$ cube with all exterior faces red; and then rearrange them to form a $3 \times 3 \times 3$ cube with all exterior faces blue; and finally, rearrange them to form a $3 \times 3 \times 3$ cube with all exterior faces white? What about the general case (n colors and n^3 $1 \times 1 \times 1$ cubes, forming an $n \times n \times n$ cube)?

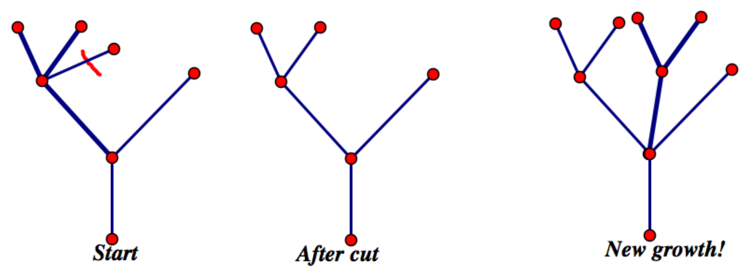
5 *Tiling with Trominos.* Define a *size-n tromino* to be the shape you get when you remove one quadrant from a $2n \times 2n$ square. In the figure below, a size-1 tromino is on the left and a size-2 tromino is on the right.



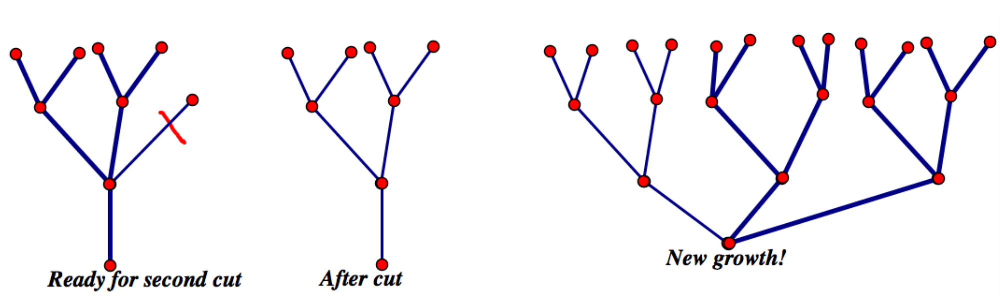
We say that a shape can be *tiled with size-1 trominos* if we can cover the entire area of the shape—*and no excess area*—with *non-overlapping* size-1 trominos. For example, it is easy to see that a 2×3 rectangle can be tiled with size-1 trominos, but a 3×3 square cannot be tiled with size-1 trominos.

Can a size-2016 tromino be tiled by size-1 trominos? What about size-2017?

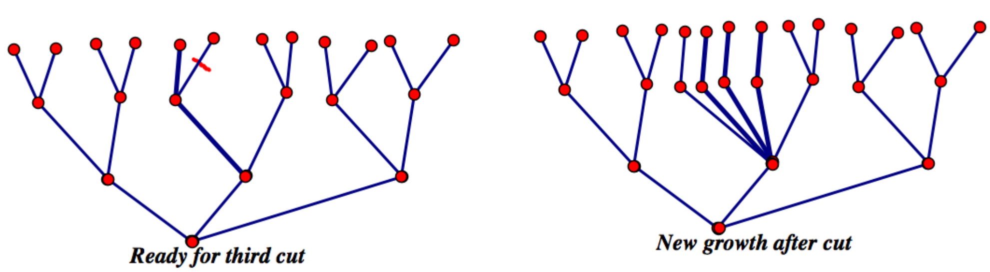
6 *Hercules vs. Hydra.* Hercules is fighting a many-headed monster, the Hydra. At each turn, he can chop off a head. But then the hydra will grow a copy of the rest of the monster, down to the branch below the cut. In the picture below, we see the cut, and the shaded lines are the part of the monster that will be duplicated. Then we see what the hydra looks like immediately after the cut. In the third picture, we see the new growth (shaded) after the duplication.



But it gets worse: after Hercules cuts a second time, the hydra will duplicate not once, but twice. And after the third cut, there will be three copies, etc. Here's a picture of what happens after a second cut. Again, the shading in the first picture shows what will be duplicated, and in the third picture, we see that two copies of this were added.



Here's a third cut. We just show two pictures. The first one indicates the cut, with shading for what will be grown back. In the second picture, we see that three copies were added!



Of course the question is, in general, can Hercules always defeat the hydra in finite time, or is it possible for the hydra to escape defeat?