SUMS AND PRODUCTS

1. *(1999 Invitational World Youth Mathematics Intercity Competition, Taiwan, Individual Contest, ♯5)* Calculate
$$1999^2 - 1998^2 + 1997^2 - 1996^2 + \cdots + 3^2 - 2^2 + 1^2.$$

2. *(1979 AMC 12, ♯11)* Find a positive integral solution to the equation:
$$\frac{1+3+5+\cdots+(2n-1)}{2+4+6+\cdots+(2n)} = \frac{115}{116}.$$
(A) 110 (B) 115 (C) 116 (D) 231 (E) the equation has no positive integral solutions

3. *(2005 Archimedes Contest, Romania, Grade 6, Final Round, ♯2)* There exists a positive integer n such that
$$\frac{8000}{19}\left(\frac{1}{1+2+3+\cdots+100} + \frac{1}{1+2+3+\cdots+101} + \cdots + \frac{1}{1+2+3+\cdots+1999}\right) = n^3.$$
Find n.

4. *(2012 Exeter Math Club Competition, Individual Accuracy Test, ♯9)* Let $f(x) = \sqrt{2x+1+2\sqrt{x^2+x}}$. Determine the value of
$$\frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \cdots + \frac{1}{f(24)}.$$

5. *(1977 AMC 12, ♯24)* Find the sum $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} + \cdots + \frac{1}{255\cdot 257}$.
(A) $\frac{127}{255}$ (B) $\frac{128}{255}$ (C) $\frac{1}{2}$ (D) $\frac{128}{257}$ (E) $\frac{129}{257}$

6. Calculate: $\frac{3^2+1}{3^2-1} + \frac{5^2+1}{5^2-1} + \frac{7^2+1}{7^2-1} + \cdots + \frac{99^2+1}{99^2-1}$.

7. Find the sum: $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \cdots + \frac{50}{1+50^2+50^4}$.

8. *(2001-2002 Mandelbrot Competition, Round 2 Individual, ♯7)* Define a sequence of numbers by
$$a_n = 3n^2 + 3n + 1,$$
so that $a_1 = 7$, $a_2 = 19$, $a_3 = 37$, and so on. Calculate $a_1 + a_2 + \cdots + a_{100}$.

9. *(1995-1996 Mandelbrot Competition, Round 2 Individual, ♯7)* A sequence a_1, a_2, a_3, ... is defined recursively by $a_1 = 1$, $a_2 = 1$, and for $k \geq 3$,
$$a_k = \frac{1}{3}a_{k-1} + \frac{1}{4}a_{k-2}.$$
Evaluate $a_1 + a_2 + a_3 + \dots$

10. *(1996 Turkey Math Contest)* Let a_n be the integer closest to $\sqrt{n}$. Find the sum
$$S = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2070}}.$$

11. *(2002 AIME I, ♯4)* Consider the sequence defined by $a_k = \dfrac{1}{k^2 + k}$ for $k \geq 1$. Given that
$$a_m + a_{m+1} + \cdots + a_{n-1} = \frac{1}{29}$$
for positive integers m and n with $m < n$, find $m + n$.

12. *(1992-1993 Mandelbrot Competition, Round 5 Individual, ♯5)* Determine the value of the infinite sum $\displaystyle\sum_{n=17}^{\infty} \frac{\binom{n}{15}}{\binom{n}{17}}$.

13. *(2002 AIME II, ♯6)* Find the integer that is closest to $1000 \displaystyle\sum_{n=3}^{10000} \frac{1}{n^2 - 4}$.

14. *(1972 AMC 12, ♯19)* The sum of the first n terms of the sequence
$$1, (1 + 2), (1 + 2 + 2^2), \dots, (1 + 2 + 2^2 + \cdots + 2^{n-1})$$
in terms of n is:

(A) 2^n (B) $2^n - n$ (C) $2^{n+1} - n$ (D) $2^{n+1} - n - 2$ (E) $n \cdot 2^n$

15. *(2006 Harvard-MIT Math Tournament, Algebra, ♯4)* Let a_1, a_2, ... be a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Find $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}$.

16. *(2001-2002 Mandelbrot Competition, Round 4 Individual, ♯7)* Let F_n be the n^{th} Fibonacci number, where as usual $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$. Find the value of the infinite sum
$$\frac{1}{3} + \frac{1}{9} + \frac{2}{27} + \cdots + \frac{F_n}{3^n} + \dots$$

17. *(1994-1995 Mandelbrot Competition, Round 1 Individual, ♯5)* It is known that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$. Given this fact, determine the exact value of

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots.$$

18. *(1998-1999 Mandelbrot Competition, Round 1 Individual, ♯8)* Define $\tau(n)=0$ if the highest power of 2 dividing n is odd, and $\tau(n)=1$ if the highest power of 2 dividing n is even. For example, $\tau(4)=1$ and $\tau(5)=1$, but $\tau(6)=0$. Compute $\sum_{n=1}^{\infty}\frac{\tau(n)}{n^2}$.

19. *(2009 AMC 12 A, ♯17)* Let $a+ar_1+ar_1^2+ar_1^3+\ldots$ and $a+ar_2+ar_2^2+ar_2^3+\ldots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is r_1+r_2?

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{1+\sqrt{5}}{2}$ (E) 2

20. *(1962 AMC 12, ♯40)* The limiting sum of the infinite series $\frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}+\ldots$ whose n^{th} term is $\frac{n}{10^n}$ is:

(A) $\frac{1}{9}$ (B) $\frac{10}{81}$ (C) $\frac{1}{8}$ (D) $\frac{17}{72}$ (E) larger than any finite quantity

21. *(1979 IMO, ♯1)* Let p and q be positive integers such that

$$\frac{p}{q}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{1318}+\frac{1}{1319}.$$

Prove that p is divisible by 1979.

22. *(1998 Gazeta Matematica, Romania)* Let

$$A=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{2011\cdot 2012}$$

and

$$B=\frac{1}{1007\cdot 2012}+\frac{1}{1008\cdot 2011}+\cdots+\frac{1}{2012\cdot 1007}.$$

Evaluate A/B.

23. *(2011 San Jose State University, Problem of the Week)* Let p and q be positive integers such that:

$$\frac{p}{q}=1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{1}{8}-\frac{2}{9}+\frac{1}{10}+\cdots+\frac{1}{1505}-\frac{2}{1506}+\frac{1}{1507}+\frac{1}{1508}.$$

Prove that p is divisible by 2011.

24. *(1989 AMC 12, ♯29)* Find $\displaystyle\sum_{k=0}^{49}(-1)^k\binom{99}{2k}$, where $\displaystyle\binom{n}{j}=\frac{n!}{j!(n-j)!}$.

(A) -2^{50} (B) -2^{49} (C) 0 (D) 2^{49} (E) 2^{50}

25. Evaluate: $\dfrac{(4\times 7+2)(6\times 9+2)(8\times 11+2)\dots(100\times 103+2)}{(5\times 8+2)(7\times 10+2)(9\times 12+2)\dots(99\times 102+2)}$.

26. *(1976 AMC 12, ♯21)* What is the smallest positive odd integer n such that the product

$$2^{1/7}2^{3/7}\dots 2^{(2n+1)/7}$$

is greater than 1000? (In the product the denominators of the exponents are all sevens, and the numerators are the successive odd integers from 1 to $2n+1$.)

(A) 7 (B) 9 (C) 11 (D) 17 (E) 19

27. *(1991 AMC 12, ♯25)* If $T_n = 1+2+3+\cdots+n$ and $P_n = \dfrac{T_2}{T_2-1}\cdot\dfrac{T_3}{T_3-1}\cdot\dfrac{T_4}{T_4-1}\cdot\cdots\cdot\dfrac{T_n}{T_n-1}$ for $n=2,3,4,\dots,$ then P_{1991} is closest to which of the following numbers?

(A) 2.0 (B) 2.3 (C) 2.6 (D) 2.9 (E) 3.2

28. *(1997-1998 Mandelbrot Competition, Round 4 Individual, ♯4)* Compute the product:

$$\frac{(1998^2-1996^2)(1998^2-1995^2)\dots(1998^2-0^2)}{(1997^2-1996^2)(1997^2-1995^2)\dots(1997^2-0^2)}.$$

29. *(1971 AMC 12, ♯32)* If $s=(1+2^{-1/32})(1+2^{-1/16})(1+2^{-1/8})(1+2^{-1/4})(1+2^{-1/2})$, then s is equal to:

(A) $\dfrac{1}{2}(1-2^{-1/32})^{-1}$ (B) $(1-2^{-1/32})^{-1}$ (C) $1-2^{-1/32}$ (D) $\dfrac{1}{2}(1-2^{-1/32})$ (E) $\dfrac{1}{2}$

30. *(2005 AIME II, ♯7)* Let $x=\dfrac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$. Find $(x+1)^{48}$.

31. *(1987 AIME, ♯14)* Compute:

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}.$$

32. *(2001-2002 Mandelbrot Competition, Round 3 Individual, ♯6)* Let F_n be the nth Fibonacci number, where as usual $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Find the value of the product

$$\prod_{k=2}^{100}\left(\frac{F_k}{F_{k-1}} - \frac{F_k}{F_{k+1}}\right),$$

leaving your answer in terms of exactly two Fibonacci numbers.

33. *(2004 AMC 12 A, ♯25)* For each integer $n \geq 4$, let a_n denote the base-n number $0.\overline{133}_n$. The product $a_4 a_5 \dots a_{99}$ can be expressed as $\frac{m}{n!}$, where m and n are positive integers and n is as small as possible. What is the value of m?

(A) 98 (B) 101 (C) 132 (D) 798 (E) 962