Computer Programming in the 18th Century
(OK, really, finite differences)

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1 Warm-up Problems

Problem 1 Find a polynomial of smallest possible degree for which $p(x)$ has $p(0) = 2$, $p(1) = 1$, $p(2) = -2$ and $p(3) = -6$. What is $p(4)$? (or you can try a simpler version of this problem in which we seek the polynomial $q(x)$ of smallest degree for whcih $q(0) = 2$, $q(1) = -1$, and $q(2) = 0$.)

Problem 2 Find a polynomial of degree at most 3 whose graph goes through the points $(-1, 5)$, $(3, 14)$, $(6, 2)$, $(11, 3)$.

Problem 3 (Modified from an ARML problem in the 90s. harder, but involves some of the same ideas) If $P(x)$ is a polynomial of degree 2014, and $P(1) = 1$, $P(2) = 1/2$, $P(3) = 1/3$, $P(4) = 1/4$, ... , $P(2015) = 1/2015$, find $P(2016)$.

Problem 4 (USAMO, 1975) If P(x) denotes a polynomial for which $P(k) = k/(k+1)$ for $k = 0, 1, 2, \ldots, n$, determine $P(n+1)$.

Problem 5 Let $p(x)$ be a polynomial with integer coefficients satisfying that $p(0)$ and $p(1)$ are odd. Show that p has no integer zeros

Problem 6 (from 2016 AIME I) Let $P(x)$ be a nonzero polynomial such that $(x-1)P(x+1) = (x+2)P(x)$ for every real x, and $(P(2))^2 = P(3)$. Then $P\left(\frac{7}{2}\right) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2 Introduction – Digression – next number in a sequence, pattern in a sequence

Problem 7 For each the following sequences, try to analyze

- What is “the” next number (or two) in “the” sequence?
- What is ~~the~~ **a** pattern that characterizes your sequence? (What types of descriptions count as a pattern?)
- Better still, Find as many patterns as you can describing the sequence.
- For each pattern, can you find other sequences that meet the same pattern? Can you characterize (in some way) the family of sequences?

(A) $3, 7, 11, 15, 19, \ldots$

(B) $13, 6, -1, -8, -15, -22, \ldots$

(C) $-1, 0, 1, 4, 9, 16, \ldots$

(D) $0, 4, 11, 21, 34, \ldots$

(E) $1, 1, 3, 13, 37, 81, \ldots$

(F) $-13, 5, 9, 5, -1, -3, 5, 29, 75, 149, 257, \ldots$

(G) $1, 2, 4, 8, 16, 32, \ldots$

(H) $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\ldots$

3 Finite Differences

Given any sequence of numbers: $a_1, a_2, a_3, a_4, \ldots$

The *sequence of differences* is given by $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$

It is convenient to write them in the following format

$$\begin{array}{ccccccccccccc} a_1 & & a_2 & & a_3 & & a_4 & & a_5 & & a_6 & & a_7 \ldots \\ & a_2 - a_1 & & a_3 - a_2 & & a_4 - a_3 & & a_5 - a_4 & & a_6 - a_5 & & a_7 - a_6 & \ldots \end{array}$$

Example:

$$\begin{array}{ccccccccccccc} -8 & & -1 & & 0 & & 1 & & 8 & & 27 & & 64 \ldots \\ & 7 & & 1 & & 1 & & 7 & & 19 & & 37 & \ldots \end{array}$$

Of course, you can take the difference of a sequence of differences, and take the difference of *that* sequence, and so on, creating an *array of differences*.

$$\begin{array}{ccccccccccccc} -8 & & -1 & & 0 & & 1 & & 8 & & 27 & & 64 \ldots \\ & 7 & & 1 & & 1 & & 7 & & 19 & & 37 & \ldots \\ & & -6 & & 0 & & 6 & & 12 & & 18 & & \ldots \\ & & & 6 & & 6 & & 6 & & 6 & & & \ldots \\ & & & & 0 & & 0 & & 0 & & & & \ldots \end{array}$$

Problem 8 Go back and do this with our example functions. Let's also try this process with the following functions: $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$. What happens? We should try to come up with some hypotheses, and maybe gather some evidence. Can you think of another family of polynomials that might be worth trying?

Problem 9 Looking at the array of differences you get for x^2 and x^3, and compare them to the array of differences for $3x^2$ and $2x^3$. Then compare those to the array differences for $3x^2 - 2x^3$. See any patterns?

Useful Notation: If we represent our sequence as a_n, we can represent the sequence of differences using the *difference operator*, Δ:

$$\Delta a_n = a_{n+1} - a_n.$$

And, we can call the difference of the difference of a sequence $\Delta(\Delta a_n)$, which can also be written (with some caution) as $\Delta^2 a_n$

4 Working backward

If I know Δa_n, can I reconstruct a_n? (At least, the terms of the sequence).

$$\begin{array}{ccccccccccccc} ? & & ? & & ? & & ? & & ? & & ? & & ? \ldots \\ & 6 & & 2 & & 0 & & 0 & & 2 & & 6 & \ldots \end{array}$$

What if I know $\Delta^2 a_n$ is the sequence $3n - 2$?

What if I know $\Delta^3 a_n$ is the constant sequence $-12, -12, -12, -12, -12, \ldots$?

5 Working diagonally

What if I know:

$$\begin{array}{ccccccccccccc}
1 & & ? & & ? & & ? & & ? & & ? & & ?\ldots \\
& -1 & & ? & & ? & & ? & & ? & & & \ldots \\
& & 2 & & ? & & ? & & ? & & & & \ldots \\
& & & -3 & & ? & & ? & & & & & \ldots \\
& & & & 0 & & 0 & & & & & & \ldots
\end{array}$$

Do I know the entire (top row) sequence? What additional assumption might allow me to complete the sequence?

Can I find a *formula* for the sequence?

Or how about:

$$\begin{array}{cccccccccccccc}
a & & ? & & ? & & ? & & ? & & ? & & ?\ldots \\
& b & & ? & & ? & & ? & & ? & & ? & \ldots \\
& & c & & c & & c & & c & & c & & c\ldots
\end{array}$$

6 The general problem and an approach to a solution

if I know values on one diagonal $d_0, d_1, d_2, \ldots d_n$ (and also that the rows below d_n is entirely 0)

$$\begin{array}{ccccccccccccc}
d_0 & & ? & & ? & & ? & & ? & & ? & & ?\ldots \\
& d_1 & & ? & & ? & & ? & & ? & & & \ldots \\
& & d_2 & & ? & & ? & & ? & & & & \ldots \\
& & & d_3 & & ? & & ? & & & & & \ldots \\
& & & & 0 & & 0 & & & & & & \ldots
\end{array}$$

Can I determine the sequence on the top row? Can I express it in a formula in terms of $d_0, d_1, \ldots, d_n$?

6.1 Repertoire method

A very useful idea **that we should verify for ourselves with examples** (and maybe even prove):

If I write the sequence a_n as the sum of two sequences b_n and c_n, then the sequence of differences of a_n is the sum of the two sequence of differences for b_n and c_n. In fact, if $a_n = j \cdot b_n + k \cdot c_n$, then

$$\Delta a_n = j \cdot \Delta b_n + k \cdot \Delta c_n$$

(we could say: the difference operator is *linear*)

Next, can we solve the general problem some special cases? In particular, let's look at diagonals that have ONE "1" on it (somewhere) and all the other entries are 0.

6.2 Pascal's Triangle

You probably already know

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

Can we see any connections to finite differences?

7 Convenient notation

It is helpful (but not universal) to use the notation for *falling powers*, that is:

$$x^{\underline{m}} = x(x-1)\cdots(x-m+1)$$

(Rising powers are similarly defined, $x^{\overline{m}} = x(x+1)\cdots(x+m-1)$, but we won't use them here.)

You may also know the expression

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\cdots(n-m+1)}{m!}$$

in connection with binomial coefficients and Pascal's triangle, but we can also consider them as polynomials in their own right:

$$\binom{x}{m} = \frac{x^{\underline{m}}}{m!} = \frac{x(x-1)\cdots(x-m+1)}{m!}$$

What is $\Delta(x^{\underline{m}})$? What is $\Delta(\binom{x}{m})$? What is $\Delta^k(x^{\underline{m}})$? $\Delta^k(\binom{x}{m})$?

The polynomial $\binom{x}{m}$ is 0 for $x = 0, 1, \ldots, m-1$ and 1 for $x = m$, (Let's verify this!) So we can see how its succession of finite differences will look. This gives a way to resurrect any polynomial from the first (well, 0th) diagonal difference sequence, solving the general problem above.

This approach also gives a nice proof of the recurrence relation:

$$p(x+n) = \binom{n}{1}p(x+n-1) - \binom{n}{2}p(x+n-2) + \ldots + (-1)^{n-1}p(x)$$

for any polynomial of degree *less* than n.

8 Problems that naturally lead to finite differences

Problem 10 Any problem where the sequence of solutions satisfies $a_{n+1} = a_n + P(n)$ where $P(n)$ is a polynomial.

- $a_{n+1} = a_n + k$
- $a_{n+1} = a_n + n$

We might need a starting point a_0 or a_1.

Problem 11 In particular, many summations

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

can be evaluated with this approach, since $S_{n+1} - S_n = \ldots$.

Problem 12 Can we evaluate:

1. $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$, $\sum_{k=1}^{n} k^3$
2. $\sum_{k=1}^{n} k \cdot (k+3)$
3. $\sum_{k=1}^{n} k^{\underline{3}}$, $\sum_{k=1}^{n} k^{\underline{4}}$
4. $\sum_{k=1}^{n} \sum_{j=1}^{k} j^2$ (th is last one came up in a problem Josh told me yesterday)

Problem 13 (Common) Into how many pieces can a pizza be divided by n straight vertical cuts? (Assume the pizza is essentially 2-dimensional – also convex. And no moving the pieces around between the cuts.)

Problem 14 Into how many pieces can a cake be cut with n straight cuts (not necessarily vertical! The point is that the cake has thickness, so now the shape is 3-dimensional and the cuts are not lines, but planes!)

Problem 15 (More repertoire method than finite differences) The polynomial equation $x^2 - x - 1 = 0$ has the two solutions $\phi = \frac{1+\sqrt{5}}{2} = 1.61803399\ldots$ and $\Phi = -0.61803399\ldots$. The recurrence relation $a_{n+1} = a_n + a_{n-1}$ has many solutions, the most famous being the fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Show that the geometric sequences $\phi^1, \phi^2, \phi^3, \ldots$ and $\Phi^1, \Phi^2, \Phi^3, \ldots$ satisfy the same recurrence relation. Verify that, if you can find a and b fo rwhich $1 = a\phi^1 + b\Phi^1$ and $1 = a\phi^2 + b\Phi^2$, then the nth Fibonacci number must be $a\phi^n + b\Phi^n$.

9 Still More Contest Problems

Problem 16 (AIME 1992) For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define ΔA to be the sequence $(a_2 - a1, a_3 - a_2, a_4 - a_3, \ldots)$, whose nth term is $a_{n+1} - a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1 and that $a_{19} = a_{92} = 0$. Find a_1.

Problem 17 (From the 1995 Polya Team Mathematics Competition) *it will be convenient for us to list the sequences in this round with initial index* 0*: that is, each sequence listed here should be considered to be of the form:* $a_0, a_1, a_2, a_3, \ldots$

(1) The sequence $1, 1, 7, 13, 55, 133, \ldots$ is an example of a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} + 6a_{n-2} \quad \text{for all } n \geq 2.$$

- **(a)** Find all geometric sequences $a_0, a_1, a_2, \ldots$ that
  - **(i)** satisfy the same recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ for all $n \geq 2$.
  - **(ii)** have the first term a_0 equal to 1.
- **(b)** For the sequence $1, 1, 7, 13, 55, 133, \ldots$ listed above, find a closed form expression for the 101^{st} term a_{100} (that is, an expression involving only simple sums, products, and exponentials, without the use of $\sum$ notation or indices).

(c) Prove that there is only one sequence of real numbers satisfying this recurrence relation with both an infinite number of positive terms and an infinite number of negative terms

(2) The sequence $0, 1, 4, 9, 16, 25, \ldots, n^2, \ldots$ is an example of a sequence that satisfies the recurrence relation

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} \quad \text{for all } n \geq 3.$$

(a) Find all geometric sequences $a_0, a_1, a_2, a_3, a_4, \ldots$ that

(i) satisfy the same recurrence relation $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ for all $n \geq 3$.

(ii) have the first term a_0 equal to 1.

(b) For the general sequence $a_0, a_1, a_2, a_3, \ldots$ satisfying the recurrence relation, find a closed form expression for a_{100} in terms of a_0, a_1, and a_2.

(c) Prove that there are no sequences of real numbers satisfying the recurrence relation with both an infinite number of positive terms and an infinite number of negative terms

(3) Prove that the sequence given by $a_0 = 2$ and, for $n \geq 1$,

$$a_n = \text{ The integer closest to } \ (5 + 2\sqrt{7})^n$$

satisfies a recurrence relation of the form $a_n = x \cdot a_{n-1} + y \cdot a_{n-2}$ for $n \geq 2$. (For partial credit, find the values for x and y.)

10 Ok, the Computer Programming in the 18th Century part

A brief outline of the history

- People were doing sophisticated mathematics and computations before there were electronic calculators and computers to evalute the functions for them.
- There are ways to compute logarithms, trig functions, square roots, etc, by hand using only arithmetic operations, but they are often complex, messy, somewhat tedious and not practical for everyday use.
- Premade mathematical tables of values were published and very commonly used by everyone with a need for computation (engineers, machinists, navigators, surveyors, architects, designers and others).
- Producing a large table of mathematical values without computing equipment involved teams of people and careful planning by mathematicians to break the task into easy (if tedious) steps.
- The first useful fact is that any "smooth" function can be approximated (at least in small interval) by a polynomial. It's much easier to evaluate a polynomial than it is a logarithm or trig function – you just have to know how to add and multiply!
- But even that is too messy do thousands of times – you don't want to have to multiply 10 digit numbers together over and over. The second useful fact is that you can evaluate a polynomial at a series of equally spaced points using only addition with the method of finite differences.
- So, to make a table of values for a function with a certain step-size:
  - A high-level mathematician would find a good polynomial to approximate a given function over a particular interval. Competing goals: want as many decimal places of accuracy as possible, but with as low a degree polynomial and over as large an interval as possible.
  - a medium-level mathematician, given the polynomial and starting point from the high-level mathematician, would compute the starting values of the finite difference for the given step size
  - teams of low-level "computers", who knew how to add and could follow instructions, would then crank away computing the values of the polynomial at successive steps.

- This process worked, but many errors crept in, both from the computations themselves or when printers typeset the handwritten manuscripts. Elaborate systems of error checking and proofreading were developed, but some errors remained.
- Charles Babbage's Difference Engine – you can see one of only two working models in the world at the Computer History Museum in Mountain View! – designed in the 1820s-1840s, would have automated the process, replacing the "computers" and the typesetters with clockwork-like gears. The high-level and medium-level mathematicians still had to do their work to set things up, but once that was done, the rest would happen without error by simply turning a crank.

11 Examples

11.1 Square Root Table

How would you compute $\sqrt{23.2}$ to eight decimal places? We can explore various methods.

And, really, our goal is to find a way to compute hundreds or thousands of values of $\sqrt{x}$, say $x = 22, 22.01, 22.02, 22.03, \ldots, 23.99, 24.00$. We don't mind doing a little work up front to get the process started, but once it starts, we'd like to keep the number and complexity of computations for each additional value in our table as low as possible.

What I'd really like is

- A polynomial whose graph is very close to the graph of $\sqrt{x}$ in the interval near my point.
  - linear interpolation from $(16, 4)$ to $(25, 5)$:
  - linear interpolation from $(20.25, 4.5)$ to $(25, 5)$.
  - quadratic interpolation through $(16, 4)$, $(20.25, 4)$, $(25, 5)$
  - fifth degree polynomial through $(20.25, 4.5), (21.16, 4.6), (22.09, 4.7), (23.04, 4.8), (24.01, 4.9), (20.25, 4.5)$
- A way to turn that polynomial into finite differences, so my computers can more easily do the calculations. (Start with $g(x) = f(22 + \frac{x}{100})$)

Computing the polynomial and differences to 10 decimal places, I get:

4.6904157598		?		?		?	...
	0.0010658825		?		?		?...
		−0.0000002421		?		?	...
			0.0000000002		?		?...

And the fifth difference, to ten decimal places, is 0!

11.2 Tangent table

Suppose we want to make a table of values of the tangent function, and today we're in the vicinity of 74 degrees. Our publisher wants the step size to be 1-minute (one sixtieth of a degree).

Our high-level mathematician tells us that the polynomial

$$P(x) = 3.6058835 + 0.244388 \cdot (x - 74.5) + 0.015394 \cdot (x - 74.5)^2 + 0.000993 \cdot (x - 74.5)^3$$

approximates $\tan x^\circ$ to within 0.000001 (10^{-6}) for all x between 74 and 75. (and within 10^{-4} for all x between 73.4 and 73.6. (and within 0.02 for all x between 70 and 78). (In "real life," we'd use a higher degree polynomial that had at least this much accuracy over a somewhat larger interval.)

This is already sort of useful, if you wanted to calculate the tangent of 74 degrees, 6 minutes, you could evaluate $P(74.1) = 3.6058835 + 0.244388 \cdot (-.4) + 0.015394 \cdot (-.4)^2 + 0.000993 \cdot (-.4)^3$, which you *could* do by hand if you had to. But we're not yet ready for our assembly line.

Our medium-level mathematician then takes this polynomial, and knowing that we want to start at 74 degrees and go up by steps of 1 minute (that is, one-sixtieth of a degree), calculates that our difference equation should start with:

3.487413875		?			?	?	...
	0.003832846		?		?	?	...
		0.000007752			?	?	...
			0.000000028		?	?	...

And now, our calculators would start doing their additions and the next sixty numbers in the top row of their computations would give us all the tangents we seek to within the desired accuracy.