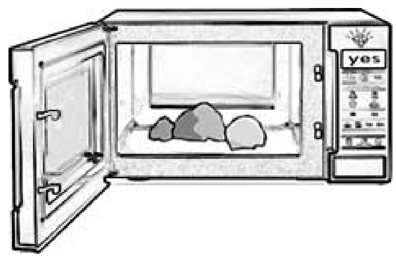
A BIT OF INFORMATION THEORY

1. You are given eight rocks. Hidden inside one of them is a valuable gem. Your goal is to determine which rock contains the gem.

You have at your disposal a gem detection machine. To use the machine, you place as many samples as you like inside its chamber, and several minutes later it indicates whether or not any of the samples contains a gem. Using the machine is costly, so you would like to minimize the number of tests required.



What strategy would you use to find the rock with the gem?

2. The following puzzle appeared in *Scientific American* magazine in 1975:

> **The poisoned glass.** “Mathematicians are curious birds,” the police commissioner said to his wife. “You see, we had all those partly filled glasses lined up in rows on a table in the hotel kitchen. Only one contained poison, and we wanted to know which one before searching that glass for fingerprints. Our lab could test the liquid in each glass, but the tests take time and money, so we wanted to make as few of them as possible by simultaneously testing mixtures of small samples from groups of glasses.” The university sent over a mathematics professor to help us. He counted the glasses, smiled and said:
>
> “Pick any glass you want, Commissioner. We’ll test it first.”
>
> “But won’t that waste a test?” I asked.
>
> “No,” he said, “it’s part of the best procedure. We can test one glass first. It doesn’t matter which one.”
>
> “How many glasses were there to start with?” the commissioner’s wife asked.
>
> “I don’t remember. Somewhere between 100 and 200.”
>
> What was the exact number of glasses?

a. What is the professor's "best procedure"?

b. Solve the puzzle.

c. How many bits of information are needed to determine the glass with the poison? Show that the professor's procedure gives that number of bits information.

d. The professor's statements, "it's part of the best procedure. We can test one glass first. It doesn't matter which one." are imprecise, and depending on the interpretation of "best," incorrect. Explain why.

3. You are given eight rocks. As in problem 1, one contains a valuable gem, and your goal is to find it using the gem detection machine. The weights of the rocks are 1, 2, 4, 5, 6, 9, 10, 13 pounds. The probability that a rock contains the gem is known to be proportional to the weight of the rock. So, for example, between two rocks, one weighing 5 pounds and one 10 pounds, the 10-pound rock is twice as likely to contain the gem.

What strategy would you use to find the rock with the gem that minimizes, on average, the number of test needed?

4. You are given eight rocks one of which contains a gem. labeled, As in the previous problems, the probability that a rock containing the gem is proportional to its weight. The rocks are labeled, from heaviest to lightest A,B,C,D,E,F,G,H. Each rock weighs an integer number of pounds and rock A weighs 13 pounds.

a. It has been determined that the sequential search, i.e. test A, then B, then C, and so on, minimizes the expected number tests needed to find the gem. What are the weights of the rocks?

b. What is the average number of tests required by the sequential search?

c. Although the sequential search minimizes, on average, the number of tests, its worst case is seven tests. Find another search that requires, on average, the same number of tests as the sequential search, but whose worst case is only four tests.